Part A

COURSE CODE: **MTH 121**

COURSE TITLE: CALCULUS

NUMBER OF UNITS: 3 Units /Compulsory

COURSE DURATION: Three hours per week

COURSE LECTURER: **ENOYOZE, Esosa.**

INTENDED LEARNING OUTCOMES

At the completion of this course, students are expected to:

1. Relate the idea of function with everyday activity
2. Solve problems related to Domain and Range of functions
3. Evaluate the values of functions
4. Construct and investigate graphs of a function.
5. Analyze the various kinds of tests for convergence
6. Relate the idea of function with everyday activities
7. Evaluate limits of several functions
8. Construct and investigate graphs of a function.
9. Investigate the nature of any function, i.e whether continuous, discontinuous or otherwise
10. Understand Differentiation as limit of rate of change of elementary function.

COURSE DETAILS:

Week 1-2: Elementary function of single real variable and their graphs, limits and the idea of continuity. Graphs of simple functions polynomial, rational, trigonometric, etc. rate of change, tangent and normal to a curve.

Week 3-5: Differentiation as limit of rate of change of elementary functions, product, quotient, function of function rules. Implicit differentiation, differentiation of trigonometric, inverse trigonometric functions and of exponential functions. Logarithmic and parametric differentiation.

Week 6-7: Stationary values of simple functions: maxima, minima and points of inflexion, area of surface revolution.

RESOURCES

• **Lecturer's Office Hours:**

• **ENOYOZE, Esosa**. Mondays: 12:30-2:30pm.

•**Course Lecture Notes: http://www.edouniversity.edu.ng/oer/maths/mth121.pdf**

• **Books:**

• Engineering Mathematics by K.A. Stroud and Dexter Booth, Macmillan Publishers. London ,6th Edition (recommended).

Schaums series on Mathematica by Don Eugene, Second Edition

Course Project:

Assignment and Problem Solving.

Using Mathematical Software to solve Problems in Calculus.

• **Exams:**

• Final, comprehensive (according to university schedule): ~ 70% of final grade

Assignments & Grading

• **Academic Honesty:** All group work should be done in teams, otherwise stated.

• General solution to problems should be discussed extensively in groups but must have individual write ups.

NO LATE HOMEWORKS ACCEPTED

• All home works are to be submitted online on the class group platform.

• All home works are due at the time stated.

• Late projects will not be accepted.

Preamble:

History of calculus

Calculus (mathematics), branch of mathematics concerned with the study of such concepts as the rate of change of one variable quantity with respect to another, the slope of a curve at a prescribed point, the computation of the maximum and minimum values of functions, and the calculation of the area bounded by curves. Evolved from algebra, arithmetic, and geometry, it is the basis of that part of mathematics called analysis.

Calculus is widely employed in the physical, biological, and social sciences. It is used, for example, in the physical sciences to study the speed of a falling body, the rates of change in a chemical reaction, or the rate of decay of a radioactive material. In the biological sciences a problem such as the rate of growth of a colony of bacteria as a function of time is easily solved using calculus. In the social sciences calculus is widely used in the study of statistics and probability.

Calculus can be applied to many problems involving the notion of extreme amounts, such as the fastest, the most, the slowest, or the least. These maximum or minimum amounts may be described as values for which a certain rate of change (increase or decrease) is zero. By using calculus it is possible to determine how high a projectile will go by finding the point at which its change of altitude with respect to time, that is, its velocity, is equal to zero. Many general principles governing the behavior of physical processes are formulated almost invariably in terms of rates of change. It is also possible, through the insights provided by the methods of calculus, to resolve such problems in logic as the famous paradoxes posed by the Greek philosopher Zeno.
The fundamental concept of calculus, which distinguishes it from other branches of mathematics and is the source from which all its theory and applications are developed, is the theory of limits of functions of variables (*see* Function)

Week 1-2:

Elementary function of single real variable and their graphs, limits and the idea of continuity. Graphs of simple functions polynomial, rational, trigonometric, etc. rate of change, tangent and normal to a curve

Topic: **Elementary function of single real variable and their graphs**

The notion of correspondence occurs frequently in everyday life. Some examples are given in the following illustrations:

1. To each book in a library there corresponds the number of pages in the book.
2. To each human being there corresponds a birth date.
3. If the temperature of the air is recorded throughout the day, then to each instant of time there corresponds a temperature.

Definition:

If a quantity y is called a 'function of x' written as f(x), it means that for every value of x(for which the function is defined) there is a corresponding value of y. x is called the argument or independent variable and y, the dependent variable of the function.

A function f from a set D to a set E is a correspondent that assigns to each element x of D exactly one element y of E.

The element x of D is the argument of f. The set D is the domain of the function.

Example

Let $g(x) = \frac{\sqrt{4+x}}{1-x}$

a. Find the Domain of g.

b. Find $g(5),\ g(-2),\ g(-a),\ and - g(a)$

Functions are common place in everyday life and show up in a variety of forms. For instance, the menu in restaurant can be considered to be a function from a set of items to a set of price. Note that f is given in a table format. Here f(meat pie)=₦200, f(biscuits)=₦150, and f(fanta)=₦100

Definition

The graph of a function f is the graph of the equation y=f(x) for x in the domain of f.

The vertical line test

The graph of a set of points in a plane is the graph of a function if every vertical line intersects the graph in at most one point.

Sketching the graph of a function

Let f(x)=$\sqrt{x-1}$

a. Sketch the graph of f.
b. Find the domain and range of f.

Example

Using a graph to find domain, range, and where a function increases or decreases.

Let f(x)=$\sqrt{9-x^2}$

a. Sketch the graph of f.
b. Find the domain and range of f.
c. Find the intervals on which f is increasing or is decreasing.

Exercise

Simplify the difference quotient

$$\frac{f(x+h)-f(x)}{h}$$

Using the function $f(x) = x^2 + 6x - 4$.

Definition

A function f is a linear function if f(x)=ax+b, where x is any real number and a and b are constants. The graph of f is the graph of y=ax+b, which by the slope – intercept, is a line with slope a and y- intercept b. Thus the graph of a linear function is a line. If a≠0, then the range of f is also ℝ.

Example

Let f(x)=2x+3.

a. Sketch the graph of f.
b. Find the domain and range of f.
c. Determine where f is increasing of decreasing.

Example

If there is a linear function such that f(-2)=5 and f(6)=3, find f(x), where x is any real number.

Exercise

In 1 to 4, if a and h are real numbers, find

a. f(a) b. f(-a) c. -f(a) d. $\frac{f(x+h)-f(x)}{h}$

1. $f(x)$ =5x-2
2. $f(x) = -x^2 + 3$
3. $f(x)$ =2x^2+3x-7
4. $f(x) = x^2 - x + 3$

Exercise

In 1 to 3, if a is a positive real number, find

a. g$\left(\frac{1}{a}\right)$
b. $\frac{1}{g(a)}$
c. g$\left(\sqrt{a}\right)$

1.,g(x)=4x^2, 2. $g(x) = 2x - 7$, 3. $g(x) = \frac{2x}{x^2+1}$

Operations on functions

Other types of functions

a. polynomials: functions like $2x^3 + 5$; $x^5 - 2x^2 + 7x + 3.$ *etc. are called* polynomials in x. These are of degrees 3,5,m respectively. Similarly, the expression

$$anx^n + \text{an} - 1x^{n-1} + \cdots + a0,\ is$$ a polynomial in x of degree n.

b. Rational functions: A rational function is function which is capable of being expressed as the quotient of two polynomials and so has the form:

$$y = \frac{anx^n + \text{an}-1x^{n-1} + \cdots + a0}{bmx^m + \text{bm}-1x^{m-1} + \cdots + b0}$$ and is defined for all variables of x for which the denominator does not vanish, n and m are degrees of the numerator and denominator polynomials respectively.

c. Transcendental functions: functions like trigonometric, hyperbolic, exponential and logarithmic are called transcendental functions e.g

$$y = \cos x,\ \cot^{-1} x, \sinh x, \cosh^{-1} x, e^x, \log x,\ etc.$$

4. Odd and even functions:

determine whether f is even, odd or neither even nor odd.

a. f(x)=$3x^4 - 2x^2 + 5$ b. $2x^5 - 7x^3 + 4x$ c. $x^3 - x^2$

Definition of composite function

The definition of composite function f∘g of two functions f and g is defined by

$$(f \cdot g)(x) = f(g(x))$$

The domain of f∘g is the set of all x in the domain of g such that g(x) is in the set of all x in the domain of g such that g(x) is in the domain of f.

A composte function is a function of a function or more precisely, a function of another function's value.

Example

Let f(x)=$x^2 - 1$ and g(x)=3x+5

a. Find (f∘g)(x) and the domain of f∘g
b. Find (g∘f)(x) and the domain of g∘f

c. Find f(g(2)) in two different ways: first using the functions f and g separately and second using the composite function f∘g

Example

Several values of two functions f and g are listed in the following table

Find (f∘g)(2), (g∘f)(2), (f∘f)(2) and (g∘g)(2).

Example

Express $y = (2x + 5)^8$ as a composite function form.

Solution

$u = 2x + 5 \; and \; y = u^8$.

Therefore, $y = u^8, \; which \; is$ a composite function form for

$y = (2x + 5)^8$

Topic: The idea of limits and continuity of a function

Limits of a function

If the limit of a function $f(x)$ is L as x approaches some number say a, then symbolically

$$\lim_{n \to a} f(x) = L$$

Note that we are concerned only with what happens to $f(x)$ as x approaches a not what happens to f(x) when x=a.

That is how

and $\lim_{x \to a^+} f(x) = L$

Is called the right- hand limit

Also as $x \longrightarrow a^+, f(x) \longrightarrow L$

And

$\lim_{x \to a^-} f(x) = L$ is called the left –hand limit.

Definition

A function f(x) is said to tend to the limit L as x—›$a.$ Iff for any arbitrarily small positive number ϵ (no matter how small), there exists another small positive number δ such that $|f(x) - L|$< ϵ , whenever 0<$|x - a|$<δ.

Theorems on limits

1. Let$f(x)$=L and$\lim_{x\to a} g(x)$ =m

Then, a. $\lim_{x\to a}[f(x) \pm g(m)]$ =L $\pm$m.

b. $\lim_{x\to a}[f(x).g(m)]$ =L .m

c. $\lim_{x\to a} \frac{f(x)}{g(x)} = \frac{L}{m}$, provided g(a)≠0

2. The limit of a constant quantity is the quantity itself.

i.e,

$\lim_{x\to a} K = k$ e.g $\lim_{x\to a} 15 = 15$

$\lim_{x\to a} \frac{\beta}{\alpha} = \lim_{x\to a} \frac{\beta_1}{\alpha_1}$

if $\alpha \simeq \alpha_1$ and $\beta \simeq \beta_1$ as x→ a, and that one of the limits exists.

Note that $\sin mx \simeq mx$ and $\tan mx \simeq mx$ as $x \to 0$

4. If evaluation and simplification of $\lim_{x\to a} \frac{f(x)}{g(x)}$ gives the indeterminate form $\frac{0}{0}$ or $\frac{\infty}{\infty}$,then we should carefully examine the quotient $\frac{f(x)}{g(x)}$ for some possible simplification and subsequently evaluate.

e.g. $\lim_{x\to a} \frac{x^2-a^2}{x-a} = \lim_{x\to a} \frac{(x+a)(x-a)}{(x-a)}$

$\lim_{x\to a}(x + a) = 2a$

In case the indeterminate form cannot be resolved by simplification, we can use the L'hospital's rule.

Exercise

1. Find the limits

a.$\lim_{x \to a}(x^3 + 5x^2 - x + 4),\ ans\ 30$

b. $\lim_{n \to \infty} \frac{2n^2+3n}{3n^2+4n},\ \ ans\ 2/3$

c. $\lim_{n \to \infty} \frac{2n^2}{n^3+1}$, ans 0

d. $\lim_{x \to \frac{\pi}{4}} \frac{1+sin2x}{1-cos4x},$ ans1

e. $\lim_{x \to \pi} \frac{tanx}{sin2x}$, ans 1/2

f. $\lim_{x \to 2} \frac{3x+6}{x^3+8}$, ans1/4

g. $\lim_{x \to \pi} \frac{sin3x}{sin3x}$, ans 5/3

. If g is the function defined by:

$$f(x) = \begin{cases} x+1, & if\ x < 2 \\ x^2 - 1, & if\ x \geq 2 \end{cases}$$

Evaluate any of the following limits which exists:

i. $\lim_{x \to 2} g(x)$

ii. $\lim_{x \to 2^+} g(x)$

iii. $\lim_{x \to 2^-} g(x)$

Lt or lim

3. $\lim_{x \to 1} \frac{x^3-1}{x-1}$

4. $\lim_{x \to 2} \frac{x-2}{x^3-3x+2}$, ans 0

5. $\lim_{x \to 3} \frac{x^2-9}{x^2-2x-3}$, ans 3/2

. $\lim_{x \to 2} \frac{3x+6}{x^3+8}$, ans 1/4

7. $\lim_{n \to \infty} \frac{n^2+1}{2n^3-1}$, ans0

8. $\lim_{x\to\pi}\frac{tanx}{sin2x}$, ans 1/2.

Continuity

The relationship between limits, continuity and a defined function is often confusing to many students at their first encounter of these concepts. A gradual simplification is prented as follows.

For example:

The function $f(x) = \begin{cases} x, \ \ for\ x < 2 \\ x, \ -3\ for\ x > 2 \end{cases}$

1. Has discontinuity at x=2 and
2. The function is not defined at x=2 even as it is shown in the equation.

As we approach 2 from the left.

$f(x) = 2$ i.e $\lim_{x\to 2^-} f(x)$=2,

And if the approach is from the right f(x)=2-3=-1

i.e $\lim_{x\to 2^+} f(x)$=-1. and since the right and left hand limits are not equal i.e.

$\lim_{x\to 2^-} f(x)$=2 $\neq$ $\lim_{x\to 2^+} f(x)$=-1,

Therefore

$\lim_{x\to 2} f(x)$ is meaningless,

Therefore the limit of the function f(x) does not exist at x=2.

Definition

A function is said to be defined at a point x=a if a value is specified for the function at x=a or the domain of definition of the function includes x=a.

For example, the function expressed in (1) is undefined at x=2, but can be made defined as follows:

The function $f(x) = \begin{cases} x, \ for\ x < 2 \\ 1, \text{ if x} = 2 \\ x, \ -3\ for\ x > 2 \end{cases}$ (2)

Though the function is still discontinuous, it is now defined at x=2 and its value is 1.

Example

Now consider the function $f(x) = \frac{x^3-1}{x-1}$ and at x=1,

$f(x) = \frac{1^3-1}{1-1}=\frac{0}{0}$ which is indeterminate meaning that f(x) does not have a value at x=1. i.e. it is undefined at x=1

Let us see if f(x) has a limit at x=1.

$\lim_{x\to1} \frac{x^3-1}{x-1} = \lim_{x\to1}(x^2 + x + 1)$

$\lim_{x\to1^-} f(x) = \lim_{x\to1^+} f(x)$=3, meaning that $\lim_{x\to1} f(x)$ *exist* and is equal to 3. we have so far established at f(x) has limit

at x=1, but it is not defined at x=1. since f(1)=$\frac{0}{0}$.

It can only be defined at x=1, if a value can be assigned to f(x) at x=1, say 2 and 3 being two of the many possibilities e.g.

$f(x) \ = \begin{cases} \frac{x^3-1}{x-1}, \ for\ x \neq 1 \\ 2, \ for\ x = 1 \end{cases}$ (a)

$$and \ \ f(x) = \begin{cases} \frac{x^3-1}{x-1}, \ for\ x \neq 1 \\ 3, \ for\ x = 1 \end{cases} \qquad (b)$$

In (a) $\lim_{x\to1} f(x)$=3, and defined value of f(x) is 2 at x=1 and since 3$\neq 2$, $f(x)$ is not continuous at x=1.

Definition

A function f(x) is said to be continuous at a point x=a if:

i. f(x) is defined at x=a.

ii.f(x) has a limit as x→ a.

i.e

$\lim_{x \to a} f(x)$=L exist.

iii. This limit is the value defined in (i)

Example

1. If the functions

a. $f(x) = \begin{cases} -1, \ for \ x \leq 2 \\ -ax + b, \ for \ -2 < x < 3 \\ 3x, \ for \ x \geq 3 \end{cases}$

b.$f(x) = \begin{cases} -x^3, \ if \ x < -1 \\ ax - b, \ if \ -1 \leq x < 1 \\ 5x, \ if \ x \geq 1 \end{cases}$

are continuous on ℝ, determine the values of a and b in each case.

Exercise

Determine whether the following functions are continuous or not at the stated point.

Also draw a graph of f(x) for values of -3≤ $x \leq 4$.

a. $f(x) = \begin{cases} x^2 - 1, \ if \ x < -2 \\ x + 5, \ if \ x \geq -2 \end{cases}$

b. $f(x) = \begin{cases} 1 - x, \ if \ x \leq 3 \\ 1 + x, \ if \ x > 3 \end{cases}$

If a function is defined by

$$f(x) = \begin{cases} a - 5x, \ \ for\ x < 2 \\ 3b + ax,\ for\ 2 \le x < 4 \\ x - b,\ \ for\ x \ge 4 \end{cases}$$

Determine the values of a and b if

$\lim_{x\to 2} f(x)\ \ and \lim_{x\to 4} f(x)$ both exist.

Next class

Differentiation

Test Wednesday on functions, limits and continuity

Mth 112 test

time allowed   30 minutes

Answer all questions

1. Define the following concept: **(a) (i)** Domain of a function**(ii)** Range of a function (iii)Composite function

.b.Represent the following interval domain using a pair of brackets

a.$|x| < 4$ **b**. $|x-4| < 1$ **c.** -1< $(x-3) \le 2$ **d.** $|x-2|^2 \le 4$

2. In 1 to 3 if a and h are real numbers, find a. $f(a)$, $b.$ $f(-a)$ $c.\frac{f(a+h)-f(a)}{h}$

i.$f(x) = -x^2 + 3$ **ii**. $f(x) = 2x^2 + 3x - 7$ **iii**. $f(x) = x^2 - x + 3$

3..Let f(x)=$x^2 - 1$ and g(x)=3x+5

a. Find (f∘g)(x) and the domain of f∘g

b. Find (g∘f)(x) and the domain of g∘f

4.Evaluate the following limits **a.** $\lim_{x\to 2}\frac{3x+6}{x^3+8}$, . ***b***. $\lim_{x\to a}\frac{x^2-a^2}{x-a}$ **c.** $\lim_{x\to 1}\frac{x^3-1}{x-1}$. **d.** $\lim_{n\to\infty}\frac{2n^2+3n}{3n^2+4n}$,

5. If g is the function defined by*:*

$$f(x) = \begin{cases} x+1, & if\ x < 2 \\ x^2 - 1, & if\ x \geq 2 \end{cases}$$

Evaluate any of the following limits which exists:

i. $\lim_{x \to 2} g(x)$ **ii**. $\lim_{x \to 2^+} g(x)$ **iii**. $\lim_{x \to 2^-} g(x)$

Week 3-5: Differentiation

Methods of first principle

Steps for differentiating from first principle are summarized below;

Step 1: Increase x by Δx i.e wherever you see x write in its place $x + \Delta x$, y is corresponding written as y+Δy on the left hand side.

Step 2: Subtract y from both sides after carrying out step 1. remember that the y being subtracted from the right hand side should be written in terms of x.

Step 3: After carrying out step 2, divide the resulting equation by Δx.

Step 4: Now take limit as $\Delta x \to 0$ i.e wherever you see Δx on the right hand side. Substitute 0 and simplify. But ratio of increments such as $\frac{\Delta y}{\Delta x} \to \frac{dy}{dx}$ as $\Delta x \to 0$.

Step 5: Your result in right hand side of step 4 is the derivative of y with respect to x

Examples

1. Find $\frac{dy}{dx}$ from first principle if (a)$y = x^2$ (b) $y = \frac{1}{x^2}$ (c) $y = x^3$ (d)$y = 2x^2 + 3x - \frac{4}{x}$

Solution

(a) $y = x^2$

Step 1: $y + \Delta y = (x + \Delta x)^2$

$$y + \Delta y = x^2 + 2x\Delta x + (\Delta x)^2$$

Step 2: $\Delta y = x^2 + 2x\Delta x + (\Delta x)^2 - y$

$$\Delta y = x^2 + 2x\Delta x + (\Delta x)^2 - x^2$$

$$\Delta y = 2x\Delta x + (\Delta x)^2$$

Step 3: $\frac{\Delta y}{\Delta x} = 2x + \Delta x$

Step 4: $\frac{dy}{dx} = \lim_{\Delta x \to 0} (2x + \Delta x) = 2x$

Step 5: $\frac{dy}{dx} = 2x$

List of Standard derivatives

	$y = f(x)$	$\frac{dy}{dx}$
1	x^n	nx^{n-1}
2	e^x	e^x
3	e^{kx}	ke^{kx}
4	a^x	$a^x. \ln a$
5	$\ln x$	$1/x$
6	$\log_a x$	$1/x. \ln a$
7	$\sin x$	$\cos x$
8	$\cos x$	$-\sin x$
9	$\tan x$	$\sec^2 x$
10	$\cot x$	$-\text{cosec}^2 x$
11	$\sec x$	$\sec x \tan x$
12	$\text{cosec}\, x$	$-\text{cosec}\, x \cot x$
13	$\sinh x$	$\cosh x$
14	$\cosh x$	$\sinh x$

Function of a function(composite rule)

Sin x is a function of x since the value of sin x depends on the value of the angle x. similarly, sin(2x+5) is a function of the angle (2x+5) since the value of the sine depends on the value of this angle.

i.e sin(2x+5) is a function of (2x+5)

But (2x+5) is itself a function of x, since its value depends on x.

i.e (2x+5) is a function of x

Sin(2x+5) is therefore a function of a function of x and such expressions are referred to generally as functions of a function.

So e^{siny} is a function of a function of y

Example

1. If $y = \cos(5x - 4)$, differentiate with respect to x

Solution

Let $U = (5x - 4), \quad \therefore y = cosu$

$$\therefore \frac{dy}{dx} = -sinu = -\sin(5x - 4).$$

But this gives us $\frac{dy}{du}, not \frac{dy}{dx}$.

To convert our result into the required derivative,we use

$$\frac{dy}{dx} = \frac{dy}{du} . \frac{du}{dx}$$

i.e, we multiply $\frac{dy}{du}$ by $\frac{du}{dx}$ to obtain $\frac{dy}{dx}$

$\frac{du}{dx}$ is found by substitution

$U = (5x - 4)$, i.e $\frac{du}{dx} = 5$

$\therefore \frac{d}{dx}\{\cos(5x - 4)\}$

$= -\sin(5x - 4) \times 5$

$= -5\sin(5x - 4)$

2. $y = e^{sinx}$, find $\frac{dy}{dx}$, Ans e^{sinx}. cosx

3. $y = \tan(5x - 4)$, $find \frac{dy}{dx}$, Ans =5$\sec^2(5x - 4)$

4. $y = (4x - 3)^5$, $find \frac{dy}{dx}$, Ans= $20(4x - 3)^4$

5.$y = \cos(x^2)$, $find \frac{dy}{dx}, Ans = -2x \sin(x^2)$

6.$y = \ln(3 - 4cosx)$, $find \frac{dy}{dx}$, Ans=$\frac{4sinx}{3-4cosx}$

7.$y = e^{3-x}$, find $\frac{dy}{dx}$, Ans= $-e^{3-x}$

7.y=cos(7x+2), $\frac{dy}{dx}$=-7sin(7x+2)

8.y=sin2x, $\frac{dy}{dx}$=2cos2x

9.y=e^{sin2x},$\frac{dy}{dx}$=2cos2x. e^{sin2x}

10.y=$\sin^2 x$,$\frac{dy}{dx}$=sin2x

Some Trigonometric identities

Sin(A+B)=SinACosB +CosASinB

Sin(A-B)=SinACosB -CosASinB

Cos(A-B)=CosACosB +SinASinB

Cos(A+B)=CosACosB –SinASinB

Sin2A=2SinACosA

Cos2A=$\cos^2 A - \sin^2 A$

Cos2A=$2\cos^2 A - 1$

Cos2A=$1-2\sin^2 A$

tan2A=$\frac{2\tan A}{1-\tan^2 A}$

$$\frac{1}{\cos^2 x} = \sec^2 x$$

$$\frac{\sin x}{\cos x} = tanx.$$

Differentiation of products

If y=uv, where u and v are functions of x, then you have that:

$$\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$$

e.g if y=$x^3.\sin 3x$

Then,

$$\frac{dy}{dx} = x^3.3\cos 3x + 3x^2 \sin 3x$$

$$=3x^2(xcos3x + sin3x).$$

To differentiate a product:

a. Put down the first, differentiate the second; plus
b. Put down the second, differentiate the first.

Examples.

1.$y = e^{2x} \ln 5x$ $\frac{dy}{dx} = e^{2x}.\frac{1}{5x}$.5+2$e^{2x}.\ln 5x$

2.y=$x^2 \tan x$; $\frac{dy}{dx} = x(x\sec^2 x + 2\tan x)$

3.y=$x^3 \sin 5x$; $\frac{dy}{dx} = x^3.5cos5x + 3x^2sin5x$

4.y=xcos $2x$, $\frac{dy}{dx} = cos2x - 2xsin2x$

5.y=$e^{5x}(3x+1)$; $\frac{dy}{dx} = e^{5x}(8+15x)$

Differentiation of quotients

In the case of quotient; if u and v are functions of x, and y=$\frac{u}{v}$,

Then $\frac{dy}{dx}$=$\frac{v\frac{du}{dx}-u\frac{dv}{dx}}{v^2}$

examples:

1. If y=$\frac{sin3x}{x+1}$, find $\frac{dy}{dx}$.

   $\frac{dy}{dx} = \frac{(x+1)3cos3x-sin3x}{(x+1)^2}$

2.y=$\frac{Inx}{e^{2x}}$, $\frac{dy}{dx}$=$\frac{\frac{1}{x}-2Inx}{e^{2x}}$

3.y=$\frac{cos2x}{x^2}$,$\frac{dy}{dx}$=$\frac{-2(xsin2x+cos2x)}{x^3}$

4.y=tanx, $\frac{dy}{dx} = \sec^2 x$

5.y=$\frac{sinx}{x^2}$, $\frac{dy}{dx} = \frac{xcosx-2sinx}{x^3}$

Logarithmic differentiation

The rules for differentiating a product or a quotient that we have revised are used when there are just two factor functions. i.e uv or $\frac{u}{v}$. Where there are more than two functions in any arrangement top or bottom, the derivation is best found by what is known as "logarithmic differentiation". It all depends on the basic fact that $\frac{d\{Inx\}}{dx}$= $\frac{1}{x}$ and that if x is replaced by a function F then $\frac{d\{InF\}}{dx}$= $\frac{1}{F}.\frac{dF}{dx}$

Bearing in mind, let us consider the case where y=$\frac{uv}{w}$, where u, v and w- and also y- are functions of x.

First take logs to the base e.

In y=In u+In v- Inw.

Now differentiate each side with respect to x, remembering that u,v, w and y are all functions of x. we get;

$$\frac{1}{y}.\frac{dy}{dx} = \frac{1}{u}\frac{du}{dx} + \frac{1}{v}\frac{dv}{dx} - \frac{1}{w}\frac{dw}{dx}$$

So to get $\frac{dy}{dx}$ by itself, we multiply across by y, to get;

$\frac{dy}{dx}=\frac{uv}{w}\left\{\frac{1}{u}\frac{du}{dx}+\frac{1}{v}\frac{dv}{dx}-\frac{1}{w}\frac{dw}{dx}\right\}$

This is a method of solving not a formula

Examples

1.If y=$\frac{x^2 sinx}{cos2x}$, find $\frac{dy}{dx}$

Solution

Take logs of both sides.

In y= In(x^2) $+ In(sinx) - In(cos2x)$

Differentiate both sides with respect to x.

$$\frac{1}{y}.\frac{dy}{dx}=\frac{1}{x^2}.2x+\frac{1}{sinx}.cosx-\frac{1}{cos2x}.(-2\sin 2x)$$

$$=\frac{2}{x}\cot x+2\tan 2x$$

$\therefore\frac{dy}{dx}=\frac{x^2 sinx}{cos2x}\left\{\frac{2}{x}\cot x+2\tan 2x\right\}$.

2.If y=$x^4e^{3x}\tan x,\ find\ \frac{dy}{dx}$.

3.y=$x^5\sin 2x\cos 4x\ ,find\ \frac{dy}{dx}$

Implicit function

If y=$x^2-4x+2,\ y$ is completely defined in terms of x, and y is called an explicit function of x.

When the relationship between x and y is more involved, it may not be possible(or desirable) to separate y completely on the LHS , e.g. xy+siny =2. In such a case as this, y is called an implicit function of x, because a relationship of the form y=f(x) is implied in the given equation.

it may still be necessary to determine the derivatives of y with respect to x and in fact this is not at all difficult. All we have to remember is that y is a function of x, even if it is difficult to see what it is. In fact, this is really an extension of our 'function of a function' routine.

x^2+y^2=25, as it stands is an example of an implicit function.

Once again, all we have to remember is that y is a function of x. so, if x^2+y^2=25, let us find $\frac{dy}{dx}$.

If we differentiate as it stands with respect to x, we get

$$2x+2y\frac{dy}{dx}=0$$

Note that we differentiate y^2 as a function squared, giving twice the function, times the derivative of the function. The rest is easy.

$2x + 2y\frac{dy}{dx} = 0$

$\therefore y\frac{dy}{dx} = -x$

$$\therefore \frac{dy}{dx} = \frac{-x}{y}$$

As you will have noticed, with an implicit function the derivative may contain both x and y.

Example 2

If $x^2 + y^2 - 2x - 6y + 5 = 0,\ find\ \frac{dy}{dx}$

solution

$$2x + 2y\frac{dy}{dx} - 2 - 6\frac{dy}{dx} = 0$$

$$(2y - 6)\frac{dy}{dx} = 2 - 2x$$

$$\therefore \frac{dy}{dx} = \frac{2-2x}{2y-6} = \frac{1-x}{y-3}$$

Example 3

If $x^2 + 2xy + 3y^2 = 4,\ find\ \frac{dy}{dx}$

Ans $\frac{dy}{dx} = \frac{-(x+y)}{(x+3y)}$

Example 4

If $x^3 + y^3 + 3xy^2 = 8$

Ans $\frac{dy}{dx} = -\frac{(x^2+y^2)}{(y^2+2xy)}$

Parametric equations

In some cases, it is more convenient to represent a function by expressing x and y separately in terms of a third independent variable, e.g $y = \cos 2t,\ x = sint.$ in this case, any value we give to t will produce a pair of values for x and y, which could if necessary be plotted and provide one point of the curve of $y = f(x)$.

The third variable, e.g. t is called a parameter, and the two expressions for x and y parametric equations. We may still need to find the derivatives of the function with repect to x.

Example

The parametric equations of a function are given as

$$y = cos2t,\ x = sint. find\ \frac{dy}{dx}$$

Solution

$$y = cos2t,\ \frac{dy}{dx} = -2sin2t.$$

$$x = sint,\ \frac{dx}{dt} = cost.$$

We can use the fact that $\frac{dy}{dx} = \frac{dy}{dt} . \frac{dt}{dx}$

$$\frac{dy}{dx} = -2sin2t. \frac{1}{cost}$$

$$= -2(2sintcost). \frac{1}{cost} = -4sint$$

Ex. 2

$$x = at^4, y = at^3; find\ \frac{dy}{dx}$$

Ans $\frac{3}{4}t^{-1}$