EDO UNIVERSITY IYAMHO

Department of Mathematics/Computer Science

MTH 221: INTRODUCTION TO DIFFERENTIAL EQUATIONS

INTRODUCTION TO DIFFERENTIAL EQUATIONS

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CHAPTER ONE

FIRST ORDER DIFFERENTIAL EQUATION

Definition1;

A differential equation can be defined as an equation comprising of independent variables $x_1, x_2, x_3, ..., x_m$ a depedent variable, y (say) together with the derivatives of y. If as in this

Definition we have more than one independent variables then the resultant equation is a Partial Differential Equation (PDE).

eg

1 $$\frac{\partial y}{\partial x_1} + 3\frac{\partial y}{\partial x_2} + 5\frac{\partial y}{\partial x_3} = f(x_1, x_2)$$

2 $$a\frac{\partial y}{\partial x_1} + b\frac{\partial y}{\partial x_2} - c\frac{\partial y}{\partial x_3} = g(x_1, x_2, x_3)$$

3 $$u(x_1, x_2)\frac{\partial y}{\partial x_1} + u(x_1, x_3)\frac{\partial y}{\partial x_2} + u(x_\partial, x_2, x_3)\frac{\partial y}{\partial x_3} = 0$$

If in the second of this example at least one of a,b and c is a function of the dependent variable or its derivative then the differential equation becomes a non linear differential equation.

On the other hand if we have a single independent variable then we have an Ordinary Differential Equation (ODE) resulting. On the other hand the order of differential equation is determined by the order of the highest derivative of the dependent variable present in the equation.

eg

1 $$a\frac{dy}{dx} + by = f(x)$$

2 $$a\frac{d^2y}{dx^2} + by = f(x)$$

3 $$a_0(x)\frac{d^my}{dx^m} + a_1(x)\frac{d^{m-1}y}{dx^{m-1}} + a_2(x)\frac{d^{m-2}y}{dx^{m-2}} + ... + a_{m-1}(x)\frac{dy}{dx} + a_my = f(x)$$

4 $$g(y)\frac{dy}{dx} + y^m = f(x)$$

5 $$\left(\frac{dy}{dx}\right)^2\frac{d^2y}{dx^2} + a\frac{dy}{dx} + by = 0$$

The example in 3 above represents the general *mth* order variable coefficients linear ODE with non homogeneous term. Examples 4 and 5 are nonlinear ODEs. In addition example 5 is homogeneous.

In this chapter we will treat the first order cases.

The general form of the first order differential equation is given as

$$\frac{dy}{dx} = F(x, y) \qquad 1.1$$

or $$A(x, y)dx + B(x, y)dy = 0 \qquad 1.2$$

1.1 SEPARABLE QUATIONS

In this case the function $F(x, y)$ in (1.1) is of the form

$$F(x, y) = g(x)h(y) \qquad 1.3$$

ie $$\frac{dy}{dx} = g(x)h(y)$$

$\Rightarrow$ $$\frac{dy}{h(y)} = g(x)dx$$

$$\int\left(\frac{dy}{h(y)}\right) = \int g(x)dx \qquad 1.4$$

The form of the final solution therefore depends on the forms of the functions $g(x)$ and $h(y)$.

Examples

1 Determine the solutions of the ODE

$$\frac{dy}{dx} = yx^2$$

Solution

Obseve that the function $F(x, y)$ on the right hand side of the diffrential equation is of separable type since it is a product of afunction of x with another function of y. Thus we may rearrange the differential equation in the new form

$$\frac{dy}{y} = x^2dx$$

$\therefore$ $$\int\frac{dy}{y} = \int x^2dx$$

ie $$\ln y = \frac{1}{3}x^3 + \alpha$$

$\Rightarrow$ $$y(x) = \exp\left[\frac{1}{3}x^3 + \alpha\right] = \exp\left(\frac{1}{3}x^3\right)\exp(\alpha)$$

ie, $$y(x) = Ae^{x^3/3}$$

2 $$\frac{dy}{dx} = 2 - 3x - 2y^2 + 3xy^2$$

Solution

Observe that $F\left(x, y\right) = 2 - 3x - 2y^2 + 3xy^2$

ie

$$F\left(x, y\right) = 2 - 3x - 2y^2 + 3xy^2$$
$$= 2 - 3x - y^2\left(2 - 3x\right)$$
$$= \left(2 - 3x\right) - y^2\left(2 - 3x\right)$$
$$= \left(2 - 3x\right)\left(1 - y^2\right)$$

Thus we have

$$\frac{dy}{dx} = \left(2 - 3x\right)\left(1 - y^2\right)$$

ie,

$$\frac{dy}{1 - y^2} = \left(2 - 3x\right)dx$$

$$\therefore \int \frac{dy}{1 - y^2} = \int \left(2 - 3x\right)dx = \beta + 2x + \frac{31}{2}x^2$$

$$Tan^{-1}y = \beta + 2x + \frac{31}{2}x^2$$

Hence, the solution to the differential equation is

$$y = Tan\left(\beta + 2x + \frac{31}{2}x^2\right)$$

1.2 Exact Differential Equations.

Recall that the general form of the first order equation may be written in the form (1.2) as

$$A\left(x, y\right)dx + B\left(x, y\right)dy = 0 \qquad 1.5$$

This differential equation is said to be exact if

$$A\left(x, y\right)dx + B\left(x, y\right)dy = dU \qquad 1.6$$

for some differentiable function $U\left(x, y\right)$

But by definition

$$dU = \frac{\partial U}{\partial x}dx + \frac{\partial U}{\partial y}dy \qquad 1.7$$

Comparing equations $(1.2.1)$ and $(1.2.2)$ we therefore have that

$$A(x,y) = \frac{\partial U}{\partial x} \text{ and } B(x,y) = \frac{\partial U}{\partial y} \qquad 1.8$$

From elementary calculus we recall that,

$$\frac{\partial^2 U}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial U}{\partial y}\right) = \frac{\partial^2 U}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial U}{\partial x}\right)$$

ie

$$\frac{\partial}{\partial x}\left(\frac{\partial U}{\partial y}\right) = \frac{\partial}{\partial y}\left(\frac{\partial U}{\partial x}\right)$$

$\Rightarrow$

$$\frac{\partial B(x,y)}{\partial x} = \frac{\partial A(x,y)}{\partial y} \qquad 1.9$$

Equation(1.9) is the necessary and sufficient condition for Equation(1.9) to be an *exact* differential equation.

Solution Technique

From $(1.2.3)$ we have that

$$A(x,y) = \frac{\partial U}{\partial x}$$

ie

$$\frac{\partial U}{\partial x} = A(x,y)$$

$$\therefore \quad U(x,y) = \int A(x,y)\,dx + g(y) \qquad 1.10$$

where $g(y)$ is an arbitrary function of y analoguos to the constant of integration in ODE.

But from the second of the equations in(1.8) we have that

$$\frac{\partial U}{\partial y} = B(x,y)$$

ie

$$\frac{\partial}{\partial y}\int\left(A(x,y)\,dx + g(y)\right) = B(x,y)$$

ie,

$$\frac{\partial}{\partial y}\int A(x,y)\,dx + g'(y) = B(x,y)$$

$\Rightarrow$

$$g'(y) = B(x,y) - \frac{\partial}{\partial y}\int A(x,y)\,dx$$

$\therefore$ $$g(y) = \int\left(B(x,y) - \frac{\partial}{\partial y}\int A(x,y)\,dx \right)dy + \beta \qquad (\beta \text{ is a constant of integration}).$$

$$= \int B(x,y)\,dy - \int\left(\frac{\partial}{\partial y}\left(\int A(x,y)\,dx\right)\right)dy + \beta$$

Thus yhe final solution to the differential equation is given as

$$U(x,y) = \int A(x,y)\,dx + \int B(x,y)\,dy - \int\left(\frac{\partial}{\partial y}\left(\int A(x,y)\,dx\right)\right)dy + \beta.$$

Examples :

Determine which of the following differential equations is exact and then solve using the method described above.:

1 $\quad x\frac{dy}{dx} + 3x + y = 0$

2 $\quad 2xy^3 + 3x^2y^2\frac{dy}{dx} = 0$

3 $\quad 2xy^2 + 2y + \left(2x^2y + 2x\right)\frac{dy}{dx} = 0$

4 $\quad \frac{xdx}{\left(x^{\to 2} + y^2\right)^{1/3}} + \frac{ydy}{\left(x^{\to 2} + y^2\right)^{2/3}} = 0$

5 $\quad \frac{dy}{dx} = \frac{ax - by}{bx - cy}$

Solution

1 Given the differential equation

$$x\frac{dy}{dx} + 3x + y = 0$$

Observe that this DE can be put in the eqivalent form

$$(3x + y)dx + xdy = 0$$

where

$$A(x, y) = 3x + y,\ B(x, y) = x$$

$$\frac{\partial A(x, y)}{\partial y} = 1 \text{ and } \frac{\partial B(x, y)}{\partial x} = 1$$

Hence by our discussion in the previous section the differential equation is exact .

Thus there exists a function $U(x, y)$ such that

$$\frac{\partial U}{\partial x} = A(x, y) = 3x + y$$

Thus

$$U(x, y) = \int (3x + y)dx + a(y)$$

$$= \frac{3}{2}x^2 + xy + a(y)$$

Recall that for exactness we also have the relation

$$\frac{\partial U}{\partial y} = B(x, y) = x$$

$\Rightarrow$

$$\frac{\partial}{\partial y}\left[\frac{3}{2}x^2 + xy + a(y)\right] = x$$

ie,

$$a'(y) + x = x$$

hence,

$$a(y) = \beta \ (\text{constant})$$

Finally, the solution to the given differential equation is

$$\frac{3}{2}x^2 + xy = \beta$$

3 $\quad 2xy^2 + 2y + (2x^2y + 2x)\frac{dy}{dx} = 0$

Solution

The DE is equivalent to

$$(2xy^2 + 2y)dx + (2x^2y + 2x)dy = 0$$

$$\frac{\partial}{\partial y}(2xy^2 + 2y) = 2,\ \frac{\partial}{\partial x}(2x^2y + 2x) = 2$$

hence, the DE is exct.

$$= h(y) - bxy + \frac{1}{2}ax^2$$

for which

$$\frac{\partial}{\partial y}\left(h(y) - bxy + \frac{1}{2}ax^2\right) = cy - bx$$

ie

$$h'(y) - bx = cy - bx$$

Thus,

$$h(y) = \int cydy + \alpha = \frac{1}{2}cy^2 + \alpha$$

The general solution to the differential equation therefore is gives as

$$\alpha + \frac{1}{2}cy^2 - bxy + \frac{1}{2}ax^2 = 0$$

1.2.1 Inexact Differential Equations.

In this section we shall study the class of Differential Equations which are not exact but can be made exact by multiplying by a suitable integrating factor $(IF) = \mu(x, y)$.

Let equation (1.2) given as

$$A(x, y)dx + B(x, y)dy = 0$$

be inexact and suppose $\mu(x, y)$ is an integrating factor

ie,

.
$$\mu(x, y)\left[A(x, y)dx + B(x, y)dy\right] = 0 \qquad (i)$$

is exact.

Then by our earlier discussion we have that

$$\frac{\partial}{\partial y}\left[\mu(x, y)A(x, y)\right] = \frac{\partial}{\partial x}\left[\mu(x, y)B(x, y)\right] \qquad (ii)$$

ie

$$\mu(x, y)\frac{\partial A(x, y)}{\partial y} + A(x, y)\frac{\partial \mu(x, y)}{\partial y} = \mu(x, y)\frac{\partial B(x, y)}{\partial x} + B(x, y)\frac{\partial \mu(x, y)}{\partial x}$$

ie,

$$A(x, y)\frac{\partial \mu(x, y)}{\partial y} - B(x, y)\frac{\partial \mu(x, y)}{\partial x} + \mu(x, y)\left[\frac{\partial A(x, y)}{\partial y} - \frac{\partial B(x, y)}{\partial x}\right] = 0 \qquad (iii)$$

This is a partial differential equation in $\mu(x, y)$ whose solution determines the exact value of the integrating factor.

ie,

Using the integrating factor $\mu(x, y) = x^m y^r$

If $\mu(x, y) = x^m y^r$ is an integrating factor to the differential equation

$$A(x, y)dx + B(x, y)dy = 0 \qquad (iv)$$

Then we must have that the new differential equation

$$x^m y^r\left[A(x, y)dx + B(x, y)dy\right]=0 \qquad (v)$$

is exact.

ie,
$$\frac{\partial}{\partial y}\left[x^m y^r A(x, y)\right]=\frac{\partial}{\partial x}\left[x^m y^r B(x, y)\right]$$

ie,

$$rx^m y^{r-1}A(x, y) + x^m y^r\frac{\partial A(x, y)}{\partial y} = rx^{m-1}y^r B(x, y) + x^m y^r\frac{\partial B(x, y)}{\partial x} \qquad (vi)$$

Now suppose the integrating factor $\mu=\mu(x)$ then condition (iii) above requires that

$$\mu(x)\left[\frac{\partial A(x, y)}{\partial y} - \frac{\partial B(x, y)}{\partial x}\right] = B(x, y)\frac{d\mu(x)}{dx} \qquad (vii)$$

ie,

$$\frac{d\mu(x)}{\mu(x)}=\frac{1}{B(x, y)}\left[\frac{\partial A(x, y)}{\partial y} - \frac{\partial B(x, y)}{\partial x}\right]dx \qquad (viii)$$

ie,

$$\int\frac{d\mu(x)}{\mu(x)}=\int\left\{\frac{1}{B(x, y)}\left[\frac{\partial A(x, y)}{\partial y} - \frac{\partial B(x, y)}{\partial x}\right]\right\}dx$$

Thus, the integrating factor $\mu(x)$ is given as

$$\mu(x) = \exp\left(\int\left\{\frac{1}{B(x, y)}\left[\frac{\partial A(x, y)}{\partial y} - \frac{\partial B(x, y)}{\partial x}\right]\right\}dx\right) \qquad (ix)$$

On the other hand if the integrating factor $\mu = \mu(y)$ then we have

$$\mu(y) = \exp\left\{\frac{1}{A(x, y)}\left[\frac{\partial B(x, y)}{\partial x} - \frac{\partial A(x, y)}{\partial y}\right]\right\} \qquad (x)$$

Example1

Obtain the general solution to the differential equation

$$\frac{dy}{dx} = -\frac{2}{y} - \frac{3y}{2x}$$

Solution

This DE is equivalent to

$$\left(4x + 3y^2\right)dx + 2xydy = 0 \qquad (i)$$

Observe that

$$\frac{\partial}{\partial y}\left(4x + 3y^2\right) = 6y \text{ and } \frac{\partial}{\partial x}\left(2xy\right) = 2y$$

Hence the DE in its present form is not exact. We therefore seek an integrating factor of the form

$$\mu\left(x, y\right) = x^m y^r . \qquad (ii)$$

$$\Rightarrow \qquad \frac{\partial}{\partial y}\left[x^m y^r \left(4x + 3y^2\right)\right] = \frac{\partial}{\partial x}\left[x^m y^r \left(2xy\right)\right]$$

ie

$$r\ x^m y^{r-1}\left(4x + 3y^2\right) + 6x^m y^{r+1} = 2\left(m+1\right)x^m y^{r+1}$$

ie

$$4rx^{m+1}y^{r-1} + 3rx^m y^{r+1} + 6x^m y^{r+1} = 2\left(m+1\right)x^m y^{r+1}$$

ie,

$$4rx^{m+1}y^{r-1} + 3\left(r+2\right)x^m y^{r+1} = 2\left(m+1\right)x^m y^{r+1} \qquad (iii)$$

Comparing coefficients of like terms on both sides of equation (iii) we thus have;

$$r = 0\ ,\ \ m = 2$$

Thus the integrating factor μ to the given differential equation is given as

$$\mu\left(x\right) = x^2 \qquad (iv)$$

Hence, there exists a potential $U\left(x, y\right)$ such that

$$\frac{\partial U}{\partial x} = 4x^3 + 3x^2y^2 \text{ and } \frac{\partial U}{\partial y} = 2x^3y \qquad (v)$$

in which the solution to the given differential equation is given as

$$U\left(x, y\right) = \text{constant.} \qquad (vi)$$

The first condition in (v) above requires that

$$U(x,y)=\int\left(4x^3+3x^2y^2\right)dx+G(y)$$

ie

$$U(x,y)=G(y)+x^4+x^3y^2$$

The second condition on the other hand requires that

$$\frac{\partial}{\partial y}\left[G(y)+x^4+x^3y^2\right]=2x^3y$$

ie,

$$G'(y)+2x^3y=2x^3y$$

Thus the solution of the differential equation is the yntegral surface defined as

$$x^4+x^3y^2=A$$

Example2

Determine the solution of the differential equation

$$3x^4-y+\left(x-4x^2y^3\right)y'=0$$

Solution

Observe that the given differential equation is equivalent to

$$\left(3x^4-y\right)dx+\left(x-4x^2y^3\right)dy=0 \qquad (i)$$

Now

$$\frac{\partial}{\partial y}\left(3x^4-y\right)=-1 \text{ and } \frac{\partial}{\partial x}\left(x-4x^2y^3\right)=1-8xy^3 \qquad (ii)$$

Thus, the differential equation in its present form is inexact and so we seek an integrating factor $\mu(x,y)$ in the form

$$\mu(x,y)=x^my^r \qquad (iii)$$

This required that

$$\frac{\partial}{\partial y}\left[x^my^r\left(3x^4-y\right)\right] \text{ and } \frac{\partial}{\partial x}\left[x^my^r\left(x-4x^2y^3\right)\right] \qquad (iv)$$

ie, $$r\,x^my^{r-1}\left(3x^4-y\right)-x^my^r=mx^{m-1}y^r\left(x-4x^2y^3\right)+x^my^r\left(1-8xy^3\right)$$

ie

$$3rx^{m+4}y^{r-1}-rx^my^r-x^my^r=mx^my^r-4mx^{m+1}y^{r+3}+x^my^r-8x^{m+1}y^{r+3}$$

ie

$$3rx^{m+4}y^{r-1}-(r+1)x^my^r=(m+1)x^my^r-4(m+2)x^{m+1}y^{r+3} \qquad (v)$$

Comparing coefficients of like terms in (v) above results in the following system of algebraic equations in m and r :

$$3r = 0$$
$$-(r+1) = (m+1)$$
$$-4(m+2) = 0$$

whose solutions are

$$r = 0 \text{ and } m = -2$$

giving the integrating factor $\mu(x) = x^2$

Thus there exiists a function $U(x, y)$ for which

$$\frac{\partial U}{\partial x} = \frac{3x^4 - y}{x^2} \text{ and } \frac{\partial U}{\partial y} = \frac{x - 4x^2 y^3}{x^2} \qquad (vi)$$

in which the integral surface $U(x, y) = A$ gives the solution to the differential equation.

From the first of condition (vi) we therefore have that

$$U(x, y) = \int \left(\frac{3x^4 - y}{x^2} \right) dx + H(y) = H(y) + \int \left(3x^2 - \frac{y}{x^2} \right) dx$$

ie,

$$U(x, y) = H(y) + x^3 + \frac{y}{x} \qquad (vii)$$

But according to the second condition in (vi) we require that

$$\frac{\partial}{\partial y}\left[H(y) + x^3 + \frac{y}{x} \right] = \frac{x - 4x^2 y^3}{x^2}$$

ie,

$$H'(y) + \frac{1}{x} = \frac{1}{x} - 4y^3 = -4y^3$$

hence

$$H(y) = -4\int y^3 dy = -y^4 + \beta$$

Thus, the solution of the differemntial equation is the integral surface

$$x^3 + \frac{y}{x} - y^4 = A$$

1.3 Homogeneous Differential Equations.

Definition:

A function $f(x, y)$ is said to be homogeneous of degree m in the variables x and y if $\exists\ \kappa$ such that

$$f(\kappa x, \kappa y) = \kappa^m f\left(x, y\right) \qquad 1.11$$

eg,

$$\text{If } f\left(x, y\right) = \left(x+y\right)^3 = x^3 + 3x^2y + 3xy^2 + y^3$$

$$f(\kappa x, \kappa y) = \left(\kappa x\right)^3 + 3\left(\kappa x\right)^2\left(\kappa y\right) + 3\left(\kappa x\right)\left(\kappa y\right)^2 + \left(\kappa y\right)^3$$

$$= \kappa^3 x^3 + 3\kappa^3 x^2 y + 3\kappa^3 xy^2 + \kappa^3 y^3$$

$$= \kappa^3\left(x^3 + 3x^2y + 3xy^2 + y^3\right)$$

$$\kappa^3 f\left(x, y\right)$$

Hence, the function is homogeneous of degree 3. A differential equation

$$\frac{dy}{dx} = F\left(x, y\right) = \frac{f\left(x, y\right)}{g\left(x, y\right)}\ ,\ g\left(x, y\right) \neq 0 \qquad 1.12$$

is said to be homogeneous if the functions $f\left(x, y\right)$ and $g\left(x, y\right)$ are homogeneous of the same degree.

Now by definition if the function $f\left(x, y\right)$ and $g\left(x, y\right)$ is homogeneous of degree m then we have that

$$\left.\begin{aligned} f\left(x, y\right) &= \kappa^{-m} f\left(\kappa x, \kappa y\right) \\ &\text{and} \\ g\left(x, y\right) &= \kappa^{-m} g\left(\kappa x, \kappa y\right) \end{aligned}\right\} \quad 1.13$$

Then

$$F\left(x, y\right) = \frac{\kappa^{-m} f\left(\kappa x, \kappa y\right)}{\kappa^{-m} g\left(\kappa x, \kappa y\right)} = \frac{\kappa^m f\left(x, y\right)}{\kappa^m g\left(x, y\right)}$$

$$= \frac{\kappa^m f\left(1, {}^{y}\!/_{x}\right)}{\kappa^m g\left(1, {}^{y}\!/_{x}\right)}$$

ie,

$$F\left(x, y\right) = \frac{f\left(1, u\right)}{g\left(1, u\right)}\ , \qquad u = {}^{y}\!/_{x}$$

Thus the original differential equation transforms into

$$\frac{dy}{dx} = \frac{f\left(1, u\right)}{g\left(1, u\right)} \qquad 1.14$$

Observe that

$$\frac{dy}{dx} = x\frac{du}{dx} + u = \frac{f(1,u)}{g(1,u)}$$

$$x\frac{du}{dx} = \frac{f(1,u)}{g(1,u)} - u = \frac{f(1,u) - ug(1,u)}{g(1,u)}$$

ie,

$$\frac{g(1,u)\,du}{f(1,u) - ug(1,u)} = \frac{dx}{x} \qquad 1.15$$

Equation (1.15) is a separable equation and we may use the method developed in section (1.1) to solve it.

Example 1

Obtain the solution to the differential equation

$$\frac{dy}{dx} = \frac{x^3 y}{x^4 + y^4}$$

Solution

Observe that the functions $x^3 y$ and $(x^4 + y^4)$ are homogeneous functions of degree 4 each, hence we may make the transformation

$$y = ux \qquad (i)$$

so that

$$\frac{dy}{dx} = x\frac{du}{dx} + u \qquad (ii)$$

Therefore the differential equation becomes

$$x\frac{du}{dx} + u = \frac{x^4 u}{x^4 + x^4 u^4} = \frac{u}{1+u^4}$$

$$x\frac{du}{dx} = \frac{u}{1+u^4} - u = -\frac{u^5}{1+u^4}$$

ie,

$$\frac{(1+u^4)\,du}{u^5} = -\frac{dx}{x} \qquad (iii)$$

$\Rightarrow$

$$\int \frac{\left(1+u^4\right)du}{u^5} = -\int \frac{dx}{x}$$

ie,

$$\int \frac{du}{u^5} + \frac{1}{5}\int \frac{5u^4}{u^5} = -\int \frac{dx}{x}$$

ie,

$$-\frac{1}{4u^4} + \frac{1}{5}\ln u^5 = -\ln x + \beta$$

$$-\frac{1}{4u^4} = \beta - \ln xu = \ln\left(\frac{A}{xu}\right)$$

ie,

$$\frac{1}{4}\left(\frac{x}{y}\right)^4 + \ln\left(\frac{A}{xu}\right) = 0$$

ie,

$$x^4 + 4y^4 \ln\left(\frac{A}{y}\right) = 0$$

*Example*2

Obtain the solution to the differential equation

$$\frac{dy}{dx} = \frac{x^2 - xy + y^2}{x^2}$$

Solution.

It is easily seen that the functions $x^2 - xy + y^2$ and x^2 are homogeneous of the same degree(2)

Hence, we make the substitution

$$y = xu \qquad (i)$$

This transforms the differential equation into

$$\frac{du}{1-2u+u^2} = \frac{dx}{x} \qquad (ii)$$

$$\int \frac{du}{1-2u+u^2} = \int \frac{dx}{x}$$

ie,

$$\frac{1}{1-u} = A\ln x$$

The solution of the differential equation is therefore the integral surface

$$\mathrm{A}x + (y-x)\ln x = 0$$

1.4 Bernoulli's Differential Equations

The differential equation

$$\frac{dy}{dx}+p(x)y(x)=Q(x)y^{m} \qquad 1.16$$

We solve (1.16) by linearizing it with the transformation

$$u=y^{1-m} \qquad 1.17$$

But

$$\frac{dy}{dx}=\frac{dy}{du}\times\frac{du}{dx}=\left(\frac{du}{dy}\right)^{-1}\frac{du}{dx}$$

ie,

$$\frac{dy}{dx}=\left[(1-m)y^{-m}\right]^{-1}\frac{du}{dx}$$

$$\Rightarrow \qquad \frac{dy}{dx}=\frac{y^{m}}{1-m}\frac{du}{dx} \qquad 1.18$$

Substituting (1.18) and (1.17) into (1.16) gives

$$\frac{y^{m}}{1-m}\frac{du}{dx}+p(x)y=Q(x)y^{m}$$

ie,

$$\frac{du}{dx}+(1-m)p(x)u=(1-m)Q(x) \qquad 1.19$$

This may now be solved using the method of integrating factor described below:

Assume an integrating factor

$$IF=\mu(x) \qquad (i)$$

ie, $\qquad \mu\left[u'+(1-m)p(x)u\right]=\left[\mu u\right]'$

ie

$$\mu u'+\mu(1-m)p(x)u=\mu u'+u\mu'$$

$\Rightarrow$

$$\frac{d\mu}{dx}=\mu(1-m)p(x)$$

ie,

$$\frac{d\mu}{\mu}=(1-m)p(x)dx$$

$$\int\frac{d\mu}{\mu}=\int(1-m)p(x)dx$$

Hence, the integrating factor $\mu(x)$ is given as

$$\mu(x)=\exp\left[\int(1-m)\,p(x)\,dx\right]$$

But

$$\left[\mu(x)u(x)\right]'=\mu(x)(1-m)Q(x)$$

$\Rightarrow$

$$\mu(x)u(x)=\int\mu(x)(1-m)Q(x)dx$$

Thus we have that

$$u(x)=\frac{1}{\mu(x)}\int\mu(x)(1-m)Q(x)dx$$

Example

Determine the solution of the differential equation

$$\frac{dy}{dx}+\frac{y}{x}=2x^3y^4$$

Solution

We observe that this is a Bernoulli's differential equation , with $m=4$

We therefore assume $u=y^{1-m}=y^{-3}$

$$\frac{dy}{dx}=\frac{dy}{du}\frac{du}{dx}=\left(\frac{du}{dx}\right)^{-1}\frac{du}{dx}=\frac{1}{3}y^4\frac{du}{dx}$$

The DE becomes

$$-\frac{1}{3}y^4\frac{du}{dx}+\frac{y}{x}=2x^3y^4$$

ie,

$$-\frac{du}{dx}+3\frac{y^{-3}}{x}=6x^3$$

ie,

$$\frac{du}{dx}-3\frac{u}{x}=-6x^3$$

Assuming an integrating factor $\mu(x)$.

$\Rightarrow$ $$\mu\left[\frac{du}{dx}-3\frac{u}{x}\right]=(\mu u)'=\mu u'+u\mu'$$

ie,

$$\frac{d\mu}{dx}=-\frac{3}{x}\mu$$

ie,

$$\frac{d\mu}{\mu} = -\frac{3}{x}dx$$

$\therefore$ $$\mu(x) = x^{-3}$$

Hence,

$$\left(x^{-3}u\right)' = 6$$

$$x^{-3}u(x) = 6x + \beta$$

$$u(x) = 6x^4 + \beta x^3$$

Finally, we have the solution

$$y^{-3} = 6x^4 + \beta x^3$$

CHAPTER TWO

HIGHER – ORDER ORDINARY DIFFERENTIAL EQUATIONS.

The most general ordinary differential equation of order *m* is given as

$$a_0(x)y^{(m)}(x)+a_1(x)y^{(m-1)}(x)+a_2(x)y^{(m-2)}(x)+.............+a_{m-1}(x)y'(x)+a_m(x)y(x)=f(x)$$

................2.1

The equation is;

(i) homogeneous if $f(x)=0$ nonhomogeneous otherwise.

(ii) constant coefficients if a_k =constant $\forall\ k$

In order to solve (2.1) we first find the *complementary function* which is the solution of the reduced equation

$$a_0(x)y^{(m)}(x)+a_1(x)y^{(m-1)}(x)+a_2(x)y^{(m-2)}(x)+.............+a_{m-1}(x)y'(x)+a_m(x)y(x)=0$$

................2.2

The most general solution of (2.2) is the linear combination of the linearlly independent functions $y_1(x), y_2(x), y_3(x),$ ...., $y_m(x)$ each of which is a solution of the DE. Thatis, the most general solution of (2.2) is given as

$$y_c(x)=A_1y_1(x)+A_2y_2(x)+A_3y_3(x)+...............+A_my_m(x) \qquad ....2.3$$

The general solution of (2.1) therefore is given as

$$y(x)=y_c(x)+y_p(x) \qquad ....2.4$$

in which $y_p(x)$ called the particular integral which is any function that satisfies the DE(2.1) and independent of $y_c(x)$.

2.1 Linear Equations with Constant coefficients.

This is the case if the coefficients in (2.1) are all constant which will be our concern in this class. Equation (2.1) therefore becomes

$$a_0y^{(m)}(x)+a_1y^{(m-1)}(x)+a_2y^{(m-2)}(x)+.............+a_{m-1}y'(x)+a_my(x)=f(x) \qquad ....2.5$$

Solving the Complementary Function

In the differential equation

We assume a solution of the form

$$y_c(x) = e^{\kappa x} \qquad 2.7$$

which on substitution gives

$$a_0\kappa^m + a_1\kappa^{m-1} + a_2\kappa^{m-2} + \ldots\ldots\ldots\ldots\ldots\ldots + a_{m-1}\kappa + a_m = 0 \qquad 2.8$$

This equation has *m* roots $\kappa_1, \kappa_2, \kappa_3, \ldots\ldots\ldots\ldots\kappa_{m-1}$ and κ_m

2.2 Second- Order Ordinary Differential Equations.

This in general takes the form

$$a_2 y''(x) + a_1 y'(x) + a_2 y(x) = f(x) \qquad 2.9$$

with the homogeneous form given as

$$a_2 y''(x) + a_1 y'(x) + a_2 y(x) = 0 \qquad 2.10$$

On assuming the solution $y_c(x) = e^{\kappa x}$ we thus have

$$\left(a_0\kappa^2 + a_1\kappa + a_2\right)e^{\kappa x} = 0$$

ie,

$$a_0\kappa^2 + a_1\kappa + a_2 = 0 \qquad 2.11$$

This is a quadratic equation called the auxiliary equation of the differential equation which has the roots

$$\kappa = \frac{-a_1 \pm \sqrt{a_1^{\,2} - 4a_0a_2}}{2a_0} \qquad 2.12$$

Assume $D = a_1^{\,2} - 4a_0a_2$ then there exists rhree distinct cases in (2.12) as follows:

Case I ; $D = 0$

This gives two repeated roots giving the solutions

$$y_1(x) = e^{\kappa_1 x}, y_2(x) = e^{\kappa_1 x} \text{ where } \kappa_1 = -\left(\frac{a_1}{2a_0}\right)$$

and in order to have linearly independent solutions we thus have $y_2(x) = xe^{\kappa_1 x}$.

Therefore the most general solution in this case is given as

$$y_c(x) = (1 + x)Ae^{\kappa_1 x} \qquad 2.13$$

Case II; $D \succ 0$

The roots of the auxiliary equation in this case are distinct and real given as

$$\kappa_1 = -\frac{a_1 + \sqrt{D}}{2a_0} \text{ and } \kappa_2 = -\frac{a_1 - \sqrt{D}}{2a_0} \qquad 2.14$$

where D is as defined earlier.

We therefore have as the solution to the differential equation

$$y_c(x) = Ae^{-(a+\omega)x} + Be^{-(a-\omega)x}$$
$$= e^{-ax}\left(Ae^{-\omega x} + Be^{\omega x}\right)$$
$$= e^{-ax}\left[A\left(Cosh\omega x - Sinh\omega x\right) + B\left(Cosh\omega x + Sinh\omega x\right)\right]$$
$$= e^{-ax}\left[\left(A+B\right)Cosh\omega x + \left(A-B\right)Sinh\omega x\right]$$

ie,

$$y_c(x) = e^{-ax}\left(A'Cosh\omega x + B'Sinh\omega x\right) \qquad 2.15$$

where

$$a = \frac{a_1}{2a_0} \text{ and } \omega = \frac{\sqrt{D}}{2a_0}$$

Case III ; $D \prec 0$.

The roots of the auxiliary equation for the differential equation in this case are two complex roots which conjugate to each other given as

$$\kappa_1 = -\frac{a_1 + i\sqrt{D}}{2a_0} \text{ and } \kappa_2 = -\frac{a_1 - i\sqrt{D}}{2a_0} \qquad 2.16$$

and the solution of the diofferential equation given as

$$y_c(x) = e^{-ax}\left(ACos\omega x + BSin\omega x\right) \qquad 2.17$$

Example 1

Obtain the complementary function of the differential equati
$y'' - 4y' + 5y = e^{2x} Sinx$

Solution

The complementary function $y_c(x)$ is the solution of the homogeneous DE

$$y'' - 4y' + 5y = 0$$

hence, we assume $y_c(x) = e^{\lambda x}$

Ubstitution of this now gives

$$(\lambda^2 - 4\lambda + 5)e^{\lambda x} = 0$$

giving the auxiliary equation \

$$\lambda^2 - 4\lambda + 5 = 0$$

whose roots are

$$\lambda_1 = 2 + i \text{ and } \lambda_2 = 2 + i$$

complementary function $y_c(x)$ is therefore

$$y_c(x) = e^{-2x}(A\sin x + BCosx)$$

2.3 Determination of Particular Integral $y_p(x)$

We shall limit ourselves to considering two methods of determining the particular integrals:

(i) Method of *Undetermined Coefficients*

(ii) Method of *Variation of Parameters*

2.3.1 Method of Undetermined Coefficients .

In this method usually we assume a solution form for the particular integral $y_p(x)$ for the general differential equation

$$a_2 y''(x) + a_1 y'(x) + a_2 y(x) = f(x) \qquad 2.18$$

depending on the form of the function $f(x)$.

Case I; if $f(x) = ae^{\gamma x}$, we assume a particular integral $y_p(x) = Ae^{\gamma x}$

Case II; if $f(x) = aSin\beta x$, we assume a particular integral $y_p(x) = ASin\beta x + BCos\beta x$

Case III; if $f(x) = a_0 x^m + a_1 x^{m-1} + a_2 x^{m-2} + ... + a_{m-1}x + a_m$

we assume a particular integral $y_p(x) = \alpha_0 x^m + \alpha_1 x^{m-1} + \alpha_2 x^{m-2} + ... + \alpha_{m-1}x + \alpha_m$

Case IV; if $f(x)$ is the sum or product of any combination of the above cases we try a solution that is of the same form.

Example.

Determine the particular integral of the differential equation

$$y'' - 4y' + 5y = e^{2x} Sinx$$

Solution.

Recall that the complementary function

$$y_c(x) = e^{2x}(ASinx + BCosx)$$

Also observe that $f(x)$ in this case is $e^{2x} Sinx$ which is a solution of the homogeneous DE.

Therefore the particular integral in this case is the function

$$y_p(x) = axe^{2x} Sinx + bxe^{2x} Cosx$$

Substitution into the DE now gives

$$(4a - 2b + 3ax - 4bx)e^{2x} Sinx + (2a + 4ax + 4b + 3bx)e^{2x} Cosx$$

$$-[(a + 2ax - bx)e^{2x} Sinx + (ax + b + 2bx)e^{2x} Cosx\ ]$$

$$+5[axe^{2x} Sinx + bxe^{2x} Cosx\ ] = e^{2x} Sinx$$

ie,

$$-2be^{2x} Sinx + 2ae^{2x} Cosx = e^{2x} Sinx$$

ie,

$$-2bSinx + 2aCosx = Sinx$$

Comparing coefficients of like terms on both sides of the equation we obtain the following system of algebraic equations in the coefficients a and b:

$$-2b = 1$$

$$2a = 0$$

These gives the result;

$$a = 0 \text{ and } b = -\frac{1}{2}$$

Therefore, the particular integral $y_p(x) = -\frac{1}{2} xe^{2x} Cosx$

The general solution of the differential equation therefore is given as

$$y(x) = e^{2x}(ASinx + BCosx) - \frac{1}{2} xe^{2x} Cosx$$

2.3.2 Method of Variation of Parameters.

Given the second order differential equation

$$a_2 y''(x) + a_1 y'(x) + a_2 y(x) = f(x) \qquad 2.19$$

With complementary function

$$y_c(x) = a_1 y_1(x) + a_2 y_2(x)$$

we assume a particular integral

$$y_p(x) = A(x)y_1(x) + B(x)y_2(x) \qquad 2.20$$

Hence,

$$y_p'(x) = \left(A'(x)y_1(x) + B'(x)y_2(x)\right) + \left(A(x)y_1'(x) + B(x)y_2'(x)\right)$$

We now assume that

$$A'(x)y_1(x) + B'(x)y_2(x) = 0 \qquad 2.21$$

Thus,

$$y_p'(x) = A(x)y_1'(x) + B(x)y_2'(x)$$

$$\therefore \qquad y_p''(x) = A(x)' y_1'(x) + B(x)' y_2'(x) + A(x)y_1''(x) + B(x)y_2''(x)$$

Substitution gives

$$a_0\left[A(x)' y_1'(x) + B(x)' y_2'(x) + A(x)y_1''(x) + B(x)y_2''(x)\right]$$
$$+ a_1\left[A(x)y_1'(x) + B(x)y_2'(x)\right] + a_2\left[A(x)y_1(x) + B(x)y_2(x)\right] = f(x)$$

ie,

$$A(x)\left[a_0 y_1''(x) + a_1 y_1'(x) + a_2 y_1(x)\right] + B(x)\left[a_0 y_2''(x) + a_1 y_2'(x) + a_2 y_2(x)\right]$$
$$+ a_0\left[A(x)' y_1'(x) + B(x)' y_2'(x)\right] = f(x) \qquad 2.22$$

Since the functions $y_1(x)$ and $y_2(x)$ are solutions of the homogeneous DE this equation thus reduces to

$$a_0\left[A(x)'\,y_1'(x)+B(x)'\,y_2'(x)\right]=f(x) \qquad 2.23$$

Combining equations $(2.3.3)$ and $(2.3.5)$ we have

$$A'(x)y_1(x)+B'(x)y_2(x)=0$$

$$A'(x)y_1'(x)+B'(x)y_2'(x)=F(x)$$

where $F(x)=\dfrac{f(x)}{a_0}$

This system of equations may be put in the form

$$\begin{pmatrix} y_1(x) & y_2(x) \\ y_1'(x) & y_2'(x) \end{pmatrix}\begin{pmatrix} A'(x) \\ B'(x) \end{pmatrix}=\begin{pmatrix} 0 \\ F(x) \end{pmatrix} \qquad 2.24$$

The matrix $W(y_1,y_2) = \begin{pmatrix} y_1(x) & y_2(x) \\ y_1'(x) & y_2'(x) \end{pmatrix}$

is called the *wronskian* of the functions $y_1(x)$ and $y_2(x)$. Thus as a condition for the system (2.24) to have non trivial solution we observe that the determinant $|W(y_1,y_2)|$. That is

$$|W(y_1,y_2)|\neq 0 \qquad 2.25$$

Putting $(2.3.6)$ in vector form we have $W(y_1,y_2)X'(x)=G(x)$

$$\therefore\ X'(x)=\frac{G(x)}{|W(y_1,y_2)|}$$

The unknowns are therefore given as

$$A'(x)=-\frac{y_2(x)G(x)}{|W(y_1,y_2)|} \quad \text{and} \quad B'(x)=\frac{y_1(x)G(x)}{|W(y_1,y_2)|} \qquad 2.26$$

Therefore, $A(x)=-\displaystyle\int\frac{y_2(x)G(x)}{|W(y_1,y_2)|}dx,\ B(x)=\int\frac{y_1(x)G(x)}{|W(y_1,y_2)|}dx$ 2.27

Finally, we have,

$$y_p(x)=y_2(x)\int\frac{y_1(x)G(x)}{|W(y_1,y_2)|}dx-y_1(x)\int\frac{y_2(x)G(x)}{|W(y_1,y_2)|}dx, \qquad 2.28$$

Example

Determine the general solution of the differential equation

$$y''+4y'+4y=x^{-2}e^{-2x}\ ,\ x\succ 0$$

Solution

The homogeneous DE is given as

$$y'' + 4y' + 4y = 0$$

By assuming a complementary function $y_c(x) = e^{\kappa x}$ we have the auxiliary equation

$$\kappa^2 + 4\kappa + 4 = 0 \Rightarrow (x+2)^2 = 0$$

ie,

$$y_1(x) = e^{-2x} \text{ and } y_2(x) = xe^{-2x}$$

Thus, we assume a particular integral

$$y_p(x) = A(x)e^{-2x} + B(x)xe^{-2x}$$

where

$$A(x) = -\int \frac{xe^{-2x}x^{-2}e^{-2x}dx}{W(y_1, y_2)} = -\int \frac{x^{-1}e^{-4x}dx}{W(y_1, y_2)}$$

$$\text{and } B(x) = \int \frac{x^{-2}e^{-4x}dx}{W(y_1, y_2)}$$

But,

$$W(y_1, y_2) = \begin{vmatrix} e^{-2x} & xe^{-2x} \\ -2e^{-2x} & (-2xe^{-2x} + e^{-2x}) \end{vmatrix} = e^{-4x}$$

ie,

$$A(x) = -\int \frac{dx}{x} = -\ln x$$

$$B(x) = \int \frac{dx}{x^2} = -\frac{1}{x}$$

Thus

$$y_p(x) = \left(-e^{-2x}\ln x - e^{-2x}\right) = -(\ln x + 1)e^{-2x}$$

The general solution of the DE therefore is

$$y(x) = \left[(\alpha + \beta x) - (1 + \ln x)\right]e^{-2x}$$

2.4 Laplace Transform Method.

Given a function $f(t)$,defined in the semi infinite interval $[0, \infty]$ then the integral

$$F(s) = \int_0^\infty e^{-st} f(t)\,dt$$

is the Laplace transform of the function $f(t)$ denoted by $L[f(t)]$. That is

$$L[f(t)] = \int_0^\infty e^{-st} f(t)\,dt \qquad 2.29$$

Properties

1 If $f(t)$ and $g(t)$ are functions whose Laplace transforms exist and α and β be any two constants then $L[\alpha f(t) + \beta g(t)] = \alpha L[f(t)] + \beta L[g(t)]$

Pr *oof*

By definition $L[\alpha f(t) + \beta g(t)] = \int_0^\infty e^{-st} (\alpha f(t) + \beta g(t)) dt$

$$= \int_0^\infty e^{-st} \alpha f(t)\,dt + \int_0^\infty e^{-st} \beta g(t)\,dt \; = \alpha \int_0^\infty e^{-st} f(t)\,dt + \beta \int_0^\infty e^{-st} g(t)\,dt$$

$$= \alpha L[f(t)] + \beta\, L[g(t)]$$

Examples

Determine the Laplace transforms of the following functions:

$t^m, e^{-at} Cosbt,$ and $e^{at} Sinhbt$

Solution

1 The Laplace transfofm of the function $t^m = \int_0^\infty e^{-st} t^m dt$

We evaluate this integral using integration by part.

Suppose $I_p = \int_0^\infty e^{-st} t^m dt = -\left[\frac{1}{s} e^{-st} t^m\right]_0^\infty + \frac{m}{s}\int_0^\infty e^{-st} t^{m-1} dt$

$$= \frac{m}{s}\int_0^\infty e^{-st} t^{m-1} dt = \frac{m}{s} I_{p-1} \; p \geq 1$$

ie, $I_p = \frac{m}{s} I_{p-1} \; p \geq 1$

Hence, $I_1 = \frac{1}{s} I_0 = = \frac{1!}{s} I_0$

$$I_2 = \frac{2}{s} I_1 = \frac{2}{s^2} I_0 == \frac{2!}{s^2} I_0$$

$$I_3 = \frac{3}{s} I_2 = \frac{6}{s^3} I_0 = \frac{3!}{s^3} I_0$$

$$I_4 = \frac{4}{s} I_3 = \frac{24}{s^4} I_0 = \frac{4!}{s^4} I_0$$

Hence, in general we have; $I_m = \frac{m}{s} I_{m-1} = \frac{m!}{s^m} I_0$

But $I_0 = \int_0^\infty e^{-st} t^0 dt = \int_0^\infty e^{-st} dt = -\left[\frac{1}{s} e^{-st}\right]_0^\infty = \frac{1}{s}$

$$\therefore \; I_m = \frac{m!}{s^{m+1}}$$

ie, $L\left[t^m\right] = \int_0^\infty e^{-st} t^m dt = \frac{m!}{s^{m+1}}$

2 To compute the Laplace transform of $e^{-at} Cosbt$.

$$L\left[e^{-at} Cosbt,\right] = \int_0^\infty e^{-st} e^{-at} Cosbtdt = \int_0^\infty e^{-(s+a)t} Cosbtdt$$

$$= -\left[\frac{e^{-(s+a)t} Cosbt}{s-a}\right]_0^\infty - \frac{b}{s-a} \int_0^\infty e^{-(s+a)t} Sinbtdt$$

$$= \frac{1}{s-a} - \frac{b}{s-a}\left[-\left[\frac{e^{-(s+a)t} Sinbt}{s-a}\right]_0^\infty + \frac{b}{s-a} \int_0^\infty e^{-(s+a)t} Cosbtdt\right]$$

$$= \frac{1}{s-a} - \left(\frac{b}{s-a}\right)^2 \int_0^\infty e^{-(s+a)t} Cosbtdt$$

ie, $\left[1 + \left(\frac{b}{s-a}\right)^2\right] \int_0^\infty e^{-(s+a)t} Cosbtdt = \frac{1}{s-a}$

ie, $\frac{(s-a)^2 + b^2}{(s-a)^2} \int_0^\infty e^{-(s+a)t} Cosbtdt = \frac{1}{s-a}$

$$\therefore \int_0^\infty e^{-(s+a)t} Cosbtdt = \frac{1}{s+a} \times \frac{(s+a)^2}{(s+a)^2+b^2} = \frac{s-a}{(s+a)^2+b^2}$$

ie,

$$L\left[e^{-at} Cosbt,\right] = \frac{s-a}{(s+a)^2+b^2}$$

3 To compute the Laplace transform of $e^{at} Sinhbt$

Solution

$$L\left[e^{at} Sinhbt\right] = \int_0^\infty e^{-(s-a)t} Sinhbtdt = \frac{1}{2}\int_0^\infty e^{-(s-a)t}\left(e^{bt} - e^{-bt}\right)dt$$

$$\frac{1}{2}\left[\int_0^\infty e^{-(s-a-b)t} dt - \int_0^\infty e^{-(s-a+b)t} dt\right]$$

$$= -\frac{1}{2}\left[\frac{e^{-(s-a-b)t}}{s-a-b} - \frac{e^{-(s-a+b)t}}{s-a+b}\right]_0^\infty = -\frac{1}{2}\left[\frac{1}{s-a+b} - \frac{1}{s-a-b}\right]$$

$$= -\frac{1}{2}\left[\frac{(s-a-b)-(s-a+b)}{(s-a-b)(s-a+b)}\right] = \frac{1}{2}\left[\frac{2b}{(s-a-b)(s-a+b)}\right]$$

ie,

$$L\left[e^{at} Sinhbt\right] = \frac{b}{(s-a-b)(s-a+b)} = \frac{b}{(s-a)^2-b^2}$$

2.4.1 Laplace Transform of Derivatives.

Given the function $f(t)$, then the transform of the derivative $\dot{f}(t)$ by definition is given as

$$L\left[\dot f(t)\right] = \int_0^\infty e^{-st}\dot f(t)\,dt$$

$$= \left[e^{-st} f(t)\right]_0^\infty + s\int_0^\infty e^{-st} f(t)\,dt$$

$$= -f(0) + s\tilde f$$

Similarly,

$$L\left[\ddot f(t)\right] = \int_0^\infty e^{-st}\ddot f(t)\,dt$$

$$= \left[e^{-st}\dot f(t)\right]_0^\infty + s\int_0^\infty e^{-st}\dot f(t)\,dt$$

$$= -\dot f(0) + sL\left[\dot f(t)\right]$$

$$= -\dot f(0) + s\left[-f(0) + s\tilde f\right]$$

ie,

$$L\left[\ddot f(t)\right] = s^2\tilde f - sf(0) - \dot f(0)$$

Furthermore, $$L\left[\dddot f(t)\right] = \int_0^\infty e^{-st}\dddot f(t)\,dt$$

$$= \left[e^{-st}\ddot f(t)\right]_0^\infty + s\int_0^\infty e^{-st}\ddot f(t)\,dt \;= -\ddot f(0) + sL\left[\ddot f(t)\right]$$

$$= -\ddot f(0) + s\left[s^2\tilde f - sf(0) - \dot f(0)\right]$$

$$= s^3\tilde f - s^2 f(0) - s\dot f(0) - \ddot f(0)$$

Hence, in general

$$L\left[f^m(t)\right] = s^m\tilde f - s^{m-1}f(0) - s^{m-2}\dot f(0) - \ldots\ldots\ldots\ldots - sf^{(m-2)}(0) - f^{(m-1)}(0)\ .$$

2.4.2 Inverse Laplace Transform.

From the above discussion on Laplace transform and its property we observe that the operator is a *linear operator*.

We then introduce the operator $L^{-1}\theta$ which is such that if

$L\left[f(t)\right]=u(s)$ then,

$$f(t)=L^{-1}\left[u(s)\right]$$

ie,

$$L^{-1}L\left[f(t)\right]=f(t)$$

ie,

$$L^{-1}L=LL^{-1}=1.$$

By vitue of the linearity of the operator L we also have that for any caonstants κ,α and β and functions $f(t)$ and $g(t)$ whose Laplace transform exist then

$$L^{-1}\left[\kappa f(t)\right]=\kappa L^{-1}\left[f(t)\right] \qquad (i)$$

$$\therefore\ L^{-1}\left[\alpha f(t)+\beta g(t)\right]=L^{-1}\left[\alpha f(t)\right]+L^{-1}\left[\beta g(t)\right]$$

ie,

$$L^{-1}\left[\alpha f(t)+\beta g(t)\right]=\alpha L^{-1}\left[f(t)\right]+\beta L^{-1}\left[g(t)\right] \qquad (ii)$$

implying thst the operator L^{-1} is also a linear operator.

The inverse Laplace transform may be found from a standard table of Laplace transfom.

For instance, from our examples above we have

1 $$L\left[e^{at}Sinhbt\right]=\frac{b}{(s-a)^2-b^2}$$

ie,

$$L^{-1}L\left[e^{at}Sinhbt\right]=L^{-1}\left[\frac{b}{(s-a)^2-b^2}\right]$$

But

$$L^{-1}L\left[f(t)\right]=f(t)$$

We therefore have

$$L^{-1}L\left[e^{at}Sinhbt\right]=L^{-1}\left[L\left[e^{at}Sinhbt\right]\right]=L^{-1}\left[\frac{b}{(s-a)^2-b^2}\right]=e^{at}Sinhbt$$

Similarly, since $L\left[e^{-at}Cosbt,\right]=\frac{s-a}{(s+a)^2+b^2}$

thus have that

$$L^{-1}\left[\frac{s-a}{(s+a)^2+b^2}\right]=e^{-at}Cosbt$$

and

$$L^{-1}\left[\frac{m!}{s^{m+1}}\right]=t^m$$

When the Laplace transform $u(s)$ of t6he function $f(t)$ is a rational function of s

ie;

$$\text{if}\quad u(s)=\frac{p(s)}{q(s)}$$

which is not a recognizable known transform of a known function then the inversion is obtained by resolution into partial fraction whose composite terms may now be easily recognizable as representing

standard transforms of known functions.

Examples

Compute the inverse Laplace transform of the following functions:

$$(i)\quad u(s)=\frac{1}{(s+\alpha)(s+\beta)}$$

$$(ii)\quad u(s)=\frac{s}{(s^2+\alpha^2)(s^2+\beta^2)}$$

$$(iii)\quad u(s)=\frac{s+\alpha}{s^2(s^2+\alpha^2)}$$

Solution.s

(i) Given $u(s)=\dfrac{1}{(s+\alpha)(s+\beta)}$

We resolve $u(s)=\dfrac{1}{(s+\alpha)(s+\beta)}$ into partial fraction thus

$$\frac{1}{(s+\alpha)(s+\beta)}=\frac{A}{(s+\alpha)}+\frac{B}{(s+\beta)}$$ where A and B are constants to be determined.

Here we have

$$\frac{A}{(s+\alpha)}+\frac{B}{(s+\beta)}=\frac{A(s+\beta)+B(s+\alpha)}{(s+\beta)(s+\alpha)}=\frac{(A+B)s+(A\beta+B\alpha)}{(s+\beta)(s+\alpha)}$$

$$\Rightarrow \quad \frac{(A+B)s+(A\beta+B\alpha)}{(s+\beta)(s+\alpha)}=\frac{1}{(s+\alpha)(s+\beta)}$$

ie, $\quad (A+B)s+(A\beta+B\alpha)=1$

Comparing coefficients of like terms on both sides gives the following system of liear equations in the unknowns A and B :

$$A+B=0$$

$$(A\beta+B\alpha)=1$$

giving the result $A=\dfrac{1}{\beta-\alpha}$, and $B=\dfrac{1}{\alpha-\beta}$

Hence

$$\frac{1}{(s+\alpha)(s+\beta)}=\frac{1}{\beta-\alpha}\left(\frac{1}{s+\alpha}-\frac{1}{s+\beta}\right)$$

ie,

$$L^{-1}\left[\frac{1}{(s+\alpha)(s+\beta)}\right]=L^{-1}\left[\frac{1}{\beta-\alpha}\left(\frac{1}{s+\alpha}-\frac{1}{s+\beta}\right)\right]$$

$$=\frac{1}{\beta-\alpha}L^{-1}\left(\frac{1}{s+\alpha}-\frac{1}{s+\beta}\right)=\frac{1}{\beta-\alpha}\left\{L^{-1}\left(\frac{1}{s+\alpha}\right)-L^{-1}\left(\frac{1}{s+\beta}\right)\right\}$$

$$=\frac{1}{\beta-\alpha}\left(e^{-\alpha t}-e^{-\beta t}\right)$$

$(ii) \quad u(s) = \dfrac{s}{(s^2+\alpha^2)(s^2+\beta^2)}$

To compute the inverse transform of $u(s) = \dfrac{s}{(s^2+\alpha^2)(s^2+\beta^2)}$

Resolving into partial fraction we have

$$\frac{s}{(s^2+\alpha^2)(s^2+\beta^2)} = \frac{As+B}{(s^2+\alpha^2)} + \frac{as+b}{(s^2+\beta^2)}$$

$$= \frac{(As+B)(s^2+\beta^2)+(as+b)(s^2+\alpha^2)}{(s^2+\alpha^2)(s^2+\beta^2)}$$

$$= \frac{(A+a)s^3+(B+b)s^2+(\beta^2 A+\alpha^2 a)s+(\beta^2 B+\alpha^2 b)}{(s^2+\alpha^2)(s^2+\beta^2)}$$

ie,

$$\frac{(A+a)s^3+(B+b)s^2+(\beta^2 A+\alpha^2 a)s+(\beta^2 B+\alpha^2 b)}{(s^2+\alpha^2)(s^2+\beta^2)} = \frac{s}{(s^2+\alpha^2)(s^2+\beta^2)}$$

$$\Rightarrow (A+a)s^3+(B+b)s^2+(\beta^2 A+\alpha^2 a)s+(\beta^2 B+\alpha^2 b) = s$$

Equating coeeficients of like terms we have the following system:

$$A+a=0$$

$$B+b=0$$

$$\beta^2 A+\alpha^2 a=1$$

$$\beta^2 B+\alpha^2 b=0$$

giving the result $A = \dfrac{1}{\beta^2-\alpha^2}$, $a = -\dfrac{1}{\beta^2-\alpha^2}$, $B=b=0$ provide $\alpha \neq \beta$

Thus the corresponding partial fraction is

$$\frac{s}{(s^2+\alpha^2)(s^2+\beta^2)} \equiv \frac{1}{\beta^2-\alpha^2}\left(\frac{s}{s^2+\alpha^2} - \frac{s}{s^2+\beta^2}\right)$$

Thus,

$$L^{-1}\left[\frac{s}{(s^2+\alpha^2)(s^2+\beta^2)}\right] = \frac{1}{\beta^2-\alpha^2}\left\{L^{-1}\left[\frac{s}{s^2+\alpha^2}\right] - L^{-1}\left[\frac{s}{s^2+\beta^2}\right]\right\}$$

$$= \frac{1}{\beta^2-\alpha^2}(Cos\alpha t - Cos\beta t)$$

$(iii)\quad u(s)=\frac{s+\alpha}{s^2\left(s^2+\alpha^2\right)}$

$$\frac{s+\alpha}{s^2\left(s^2+\alpha^2\right)}\equiv\frac{As+B}{\left(s^2+\alpha^2\right)}+\frac{as+b}{s^2}$$

$$=\frac{\left(As+B\right)s^2+\left(s^2+\alpha^2\right)\left(as+b\right)}{s^2\left(s^2+\alpha^2\right)}=\frac{\left(A+a\right)s^3+\left(B+b\right)s^2+\alpha^2as+\alpha^2b}{s^2\left(s^2+\alpha^2\right)}$$

ie,

$$\frac{\left(A+a\right)s^3+\left(B+b\right)s^2+\alpha^2as+\alpha^2b}{s^2\left(s^2+\alpha^2\right)}\equiv\frac{s+\alpha}{s^2\left(s^2+\alpha^2\right)}$$

$\therefore\quad \left(A+a\right)s^3+\left(B+b\right)s^2+\alpha^2as+\alpha^2b=s+\alpha$

Thus

$$A+a=0$$
$$B+b=0$$
$$\alpha^2a=1$$
$$\alpha b=1$$

giving

$$A=-\frac{1}{\alpha^2},\ B--\frac{1}{\alpha},\ a=\frac{1}{\alpha^2}\ \text{and}\ b=\frac{1}{\alpha}$$

Thus the partial fraction corresponding to the given function is

$$\frac{1}{\alpha^2}\left(\frac{s+\alpha}{s^2}-\frac{s+\alpha}{s^2+\alpha^2}\right)=\frac{1}{\alpha^2}\left(\frac{1}{s}+\frac{\alpha}{s^2}-\frac{s}{s^2+\alpha^2}-\frac{s}{s^2+\alpha^2}\right)$$

ie,

$$L^{-1}\left[\frac{s+\alpha}{s^2\left(s^2+\alpha^2\right)}\right]=\frac{1}{\alpha^2}L^{-1}\left(\frac{1}{s}+\frac{\alpha}{s^2}-\frac{s}{s^2+\alpha^2}-\frac{\alpha}{s^2+\alpha^2}\right)$$

$$\frac{1}{\alpha^2}\left\{L^{-1}\left[\frac{1}{s}\right]+\alpha L^{-1}\left[\frac{1}{s^2}\right]-L^{-1}\left[\frac{s}{s^2+\alpha^2}\right]-L^{-1}\left[\frac{\alpha}{s^2+\alpha^2}\right]\right\}$$

$$=\frac{1}{\alpha^2}+\frac{t}{\alpha}-\frac{1}{\alpha^2}\left(Cos\alpha t+Sin\alpha t\right)$$

ie,

$$f\left(t\right)=\frac{1}{\alpha^2}\left[1+\alpha t-Cos\alpha t-Sin\alpha\right]$$

2.4.3 Solution of Initial Value Problems.

The differential equation

$$a_2\ddot{x}(t)+a_1\dot{x}(t)+a_2x(t)= f(t) \qquad (2.30)$$

together with the auxiliary condition

$$x(t_0)=a,\, \dot{x}(t_0)=b \qquad (2.31)$$

is called an *initial value problem* with the auxiliary condition (2.31) refered to as the initial conditions.

It is often convenient to solve (2.30) by taking its Laplace transform which is later inverted to obtain the unknown function $x(t)$.

Examples.

Determine the solutions of the following initial value problems:

1 $\ddot{x}(t)+4x(t)=Sin3t$, $x(0)=2, \dot{x}(0)=-1$

2 $\ddot{x}(t)+4\dot{x}(t)+8x(t)=Cos2t$, $x(0)=2, \dot{x}(0)=1$

3 $\ddot{x}(t)+2y(t)+x(t)=0$

$2\ddot{y}(t)-3\ddot{x}(t)+4y(t)-3x(t)=0$,$x(0)=1=y(0), \dot{x}(0)=0, \dot{y}(0)=-1$

Solution

1 $\ddot{x}(t)+4x(t)=Sin3t$, $x(0)=2, \dot{x}(0)=-1$

Taking the Laplace transform of the diferential equation we have,

$$L\left[\ddot{x}(t)+4x(t)\right]=L\left[Sin3t\right]$$

ie,

$$L\left[\ddot{x}(t)\right]+4L\left[x(t)\right]=L\left[Sin3t\right]$$

Suppose $L\left[x(t)\right]=u(s)$ then we have

$$s^2u-sx(0)-\dot{x}(0)+4u=\frac{3}{s^2+9}$$

ie,

$$s^2u-2s+1+4u=\frac{3}{s^2+9}$$

ie,

$$\left(s^2+4\right)u=\frac{3}{s^2+9}+2s-1=\frac{3+\left(2s-1\right)\left(s^2+9\right)}{s^2+9}$$

ie,

$$u=\frac{2s^3-s^2+18s-6}{\left(s^2+4\right)\left(s^2+9\right)}$$

The solution of the differential equation $x\left(t\right)$ is given as

$$x\left(t\right)=L^{-1}\left[u\right]$$

ie,

$$x\left(t\right)=L^{-1}\left[\frac{2s^3-s^2+18s-6}{\left(s^2+4\right)\left(s^2+9\right)}\right]$$

Resolving $\frac{2s^3-s^2+18s-6}{\left(s^2+4\right)\left(s^2+9\right)}$ into partial frctions we have

$$\frac{2s^3-s^2+18s-6}{\left(s^2+4\right)\left(s^2+9\right)}=\frac{As+B}{s^2+4}+\frac{as+b}{s^2+9}=\frac{\left(As+B\right)\left(s^2+9\right)+\left(as+b\right)\left(s^2+4\right)}{\left(s^2+4\right)\left(s^2+9\right)}$$

ie,

$$\frac{2s^3-s^2+18s-6}{\left(s^2+4\right)\left(s^2+9\right)}=\frac{\left(As+B\right)\left(s^2+9\right)+\left(as+b\right)\left(s^2+4\right)}{\left(s^2+4\right)\left(s^2+9\right)}$$

Thus,

$$\left(A+a\right)s^3+\left(B+b\right)s^2+\left(9A+4a\right)s+\left(9B+4b\right)=2s^3-s^2+18s-6.$$

from which we have the following system of algebraic equations:

$$A+a=2$$
$$B+b=-1$$
$$9A+4a=18$$
$$9B+4b=-6$$

with solutions

$$A=2, B=-\frac{2}{5}, A=0 \text{ and } b=-\frac{3}{5}$$

$$\therefore \qquad u(s)=\frac{1}{5}\left(\frac{10s-2}{s^2+4}-\frac{3}{s^2+9}\right)$$

Hence,

$$x(t)=L^{-1}\left[u(s)\right]=\frac{1}{5}L^{-1}\left[\frac{10s-2}{s^2+4}-\frac{3}{s^2+9}\right]$$

$$=\frac{1}{5}L^{-1}\left[\frac{10s}{s^2+4}-\frac{2}{s^2+4}-\frac{3}{s^2+9}\right]$$

$$=\frac{1}{5}L^{-1}\left[5\left(\frac{2s}{s^2+4}\right)-\frac{2}{s^2+4}-\frac{3}{s^2+9}\right]$$

$$=2L^{-1}\left[\frac{s}{s^2+4}\right]-\frac{1}{5}L^{-1}\left[\frac{2}{s^2+4}\right]-\frac{1}{5}L^{-1}\left[\frac{3}{s^2+9}\right]$$

ie,

$$x(t)=2Cos2t-\frac{1}{5}\left(Sin2t+Sin3t\right)$$

2 $\ddot{x}(t)+4\dot{x}(t)+8x(t)=Cos2t$, $x(0)=2,\dot{x}(0)=1$

Solution

Suppose the Laplace transform of $x(t)$ is $y(s)$ then

$$s^2y-sx(0)-\dot{x}(0)+4\left[sy-x(0)\right]+8y=\frac{s}{s^2+4}$$

ie, $\quad s^2y-2s-1+4\left[sy-2\right]+8y=\frac{s}{s^2+4}$

$$\Rightarrow \quad \left(s^2+4s+8\right)y=\frac{s}{s^2+4}+2s+9=\frac{s+\left(s^2+4\right)\left(2s+9\right)}{s^2+4}$$

ie,

$$y(s)=\frac{2s^3+9s^2+9s+36}{\left(s^2+4s+8\right)\left(s^2+4\right)}=\frac{As+B}{s^2+4s+8}+\frac{Ds+E}{s^2+4}$$

ie,

$$\frac{\left(As+B\right)\left(s^2+4\right)+\left(s^2+4s+8\right)\left(s^2+4\right)}{\left(s^2+4s+8\right)\left(s^2+4\right)}=\frac{2s^3+9s^2+9s+36}{\left(s^2+4s+8\right)\left(s^2+4\right)}$$

$\Rightarrow$

$$(A+D)s^3+(B+4D+E)s^2+(4A+8D+4E)s+(4B+8E)=2s^3+9s^2+9s+36$$

This gives the linear system

$$A+D=2$$
$$B+4D+E=9$$
$$4A+8D+4E=9$$
$$B+2E=9$$

giving $A=\frac{39}{20},\ B=\frac{43}{5},\ D=\frac{1}{20},\ E=\frac{1}{5}$

ie, $$y(s)=\frac{1}{20}\left(\frac{39s+172}{s^2+4s+8}+\frac{s+4}{s^2+4}\right)=\frac{1}{20}\left[\frac{39s+172}{(s+2)^2+4}+\frac{s+4}{s^2+4}\right]$$

$$=\frac{1}{20}\left[39\frac{s+2}{(s+2)^2+4}+47\frac{2}{(s+2)^2+4}+\frac{s}{s^2+4}+2\frac{2}{s^2+4}\right]$$

$\therefore$ $$x(t)=\frac{1}{20}\left(39e^{-2t}Cos2t+47e^{-2t}Sin2t+Cos2t+2Sin2t\right)$$

3 $\ddot{x}(t)+2\dot{y}(t)+x(t)=0$

$2\ddot{y}(t)-3\ddot{x}(t)+4y(t)-3x(t)=0$ $,x(0)=1=y(0),\dot{x}(0)=0,\dot{y}(0)=-1$

Solution

Let $L[x(t)]=u(s)$ and $L[y(x)]=U(s)$ Then,

$$s^2u-sx(0)-\dot{x}(0)+2(sU-y(0))+u=0$$

$$2\left[s^2U-sy(0)-\dot{y}(0)\right]-3\left[s^2u-sx(0)-\dot{x}(0)\right]+4U-3u=0$$

ie, $$(s^2+1)u+2sU=s+2$$

$$2(s^2+2)U-3(s^2+1)u=-(s+2)$$

ie, $$(s^2+3s+2)U=(s+2)$$

$\therefore$ $$U=\frac{s+2}{s^2+3s+2}=\frac{1}{s+1}$$ and

$$u=\frac{s^2+s+2}{(s+1)(s^2+1)}=\frac{1}{s+1}+\frac{1}{s^2+1}$$

Hence,

$$x(t)=e^{-t}+Sint$$

$$y(t)=e^{-t}$$

CHAPTER THREE

INTRODUCTION TO PARTIAL DIFFERENTIAL EQUATIONS.

We shall treat briefly in this chapter the simple cases of:

1. Elliptic Differential Equation- Laplace Equation
2. Parabolic Differential Equation-Heat Equation
3. Hyperbolic Differential Equation-Wave Equation

using the method of separation of variables.

3.1 Elliptic Differential Equation

The potential in a plane within a region $\prod : 0 \le x \le a ; 0 \le y \le b$ is known to be governed by the laplace differential equation

$\nabla^2 U = 0 \ ; \ 0 \prec x \prec a$

$U(0, y) = 0 = U(a, y) \ ; \ 0 \le y \le b$

$U(x, 0) = 0; \ U(x, b) \ ; \ 0 \le x \le a$

Solution

Observe that the differential equation is equivalent to

$$\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} = 0 \qquad (i)$$

We thus assume a solution of the form

$$U(x, y) = \mathrm{X}(x)\mathrm{Y}(y) \neq 0 \qquad (ii)$$

giving

$$\frac{\partial U}{\partial x} = \mathrm{Y}(y)\frac{dX}{dx}, \quad \frac{\partial^2 U}{\partial x^2} = \mathrm{Y}(y)\frac{d^2 X}{dx^2}, \quad \frac{\partial U}{\partial y} = X(x)\frac{dY}{dy}, \quad \frac{\partial^2 U}{\partial y^2} = X(x)\frac{d^2 Y}{dy^2}, \qquad (iii)$$

Substitution gives

$$\mathrm{Y}(y)\frac{d^2X}{dx^2}+X(x)\frac{d^2Y}{dy^2}=0 \qquad (iv)$$

Dividing through by $U(x,y)=\mathrm{X}(x)\mathrm{Y}(y)\neq 0$ we have

$$\frac{1}{X(x)}\ \frac{d^2X}{dx^2}=-\frac{1}{Y(y)}\frac{d^2Y}{dy^2} \qquad (v)$$

Observe that the LHS of (v) is a function of x only while the RHS is a function of function of y only thus they must be equal to a constant.

ie,

$$\frac{1}{X(x)}\ \frac{d^2X}{dx^2}=-\frac{1}{Y(y)}\frac{d^2Y}{dy^2}=-\lambda^2 \qquad (v)$$

This results in the following uncoupled ordinary differential equation:

$$\frac{d^2X}{dx^2}+\lambda^2X(x)=0$$

$$(vi)$$

$$\frac{d^2Y}{dy^2}-\lambda^2Y(y)=0$$

with solutions

$$X(x)=ASin\lambda x+BCos\lambda x\ \ ;\ \ Y(y)=PSinh\lambda y+QCosh\lambda y$$

Applying the Boundary Conditions

From the first set of Boundary Conditions :

$$U(0,y)=0\Rightarrow X(0)Y(y)=0$$

$\Rightarrow$

$$X(0)=0\ \because\ Y(y)\neq 0$$

This therefore requires that $B=0$

Thus

$$X(x)=ASin\lambda x$$

Similarly,

$$U(a,y)=0\Rightarrow X(0)=0$$

ie,

$$ASin\lambda a=0\Rightarrow Sin\lambda a=0$$

Hence,

$$\lambda_m a = Sin^{-1} 0 = m\pi \quad , \quad m \in \mathbb{Z}$$

ie,

$$\lambda_m = \frac{m\pi}{a}$$

This therefore gives

$$X_m(x) = A_m Sin \frac{m\pi}{a} x \qquad (vii)$$

Similarly, from the first of the second set of boundary conditions we have

$$U(x,0) = 0 \Rightarrow X(x)Y(0) = 0$$

ie

$$X(x)Y(0) = 0$$

This thus requires that

$$Y(0) = 0 \quad , \quad \because X(x) \neq 0$$

Thus we have

$$Y(y) = PSinh\lambda y$$

ie,

$$Y_m(y) = P_m Sinh\lambda_m y \qquad (viii)$$

Combining (vii) and $(viii)$ we therefore have

$$U_m(x,y) = D_m Sin \frac{m\pi}{a} x Sinh \frac{m\pi}{a} y$$

By the linearity of the problem thus the most general solution $U(x,y)$ is given as

$$U(x,y) = \sum_{m=1}^{\infty} U_m(x,y)$$

ie,

$$U(x,y) = \sum_{m=1}^{\infty} D_m Sin \frac{m\pi x}{a} Sinh \frac{m\pi y}{a} \qquad (ix)$$

The last of the boundary condition gives

$$U(x,b) = U_0$$

ie,

$$\sum_{m=1}^{\infty} D_m Sin \frac{m\pi x}{a} Sinh \frac{m\pi b}{a} = U_0 \qquad (x)$$

Multiplying through $(*)$ by $Sin\frac{p\pi}{a}x$ and integrate with respect to x from 0 to a we have

$$\int_0^a \sum_{m=1}^{\infty} D_m Sin\frac{m\pi}{a}xSinh\frac{m\pi}{a}bSin\frac{p\pi}{a}xdx = \int_0^a U_0 Sin\frac{p\pi}{a}xdx$$

ie,

$$\sum_{m=1}^{\infty} D_m Sinh\frac{m\pi b}{a} \int_0^a Sin\frac{m\pi x}{a}Sin\frac{p\pi x}{a}dx = \int_0^a U_0 Sin\frac{p\pi x}{a}dx$$

$\Rightarrow$

$$D_p Sinh\frac{p\pi b}{a}\int_0^a Sin^2\left(\frac{p\pi x}{a}\right)dx = \int_0^a U_0 Sin\frac{p\pi x}{a}dx \quad (\text{orthogonalty of Sine function})$$

ie,

$$\frac{D_p}{2}Sinh\frac{p\pi b}{a}\int_0^a \left[1 - Sin\left(\frac{2p\pi x}{a}\right)\right]dx = \int_0^a U_0 Sin\frac{p\pi x}{a}dx$$

ie,

$$\frac{bD_p}{2}Sinh\frac{p\pi b}{a} = \int_0^a U_0 Sin\frac{p\pi x}{a}dx = 0 = \frac{aU_0}{p\pi}\left[Cos\frac{p\pi x}{a}\right]_0^a$$

ie,

$$\frac{bD_p}{2}Sinh\frac{p\pi b}{a} = \frac{aU_0}{p\pi}\left[(-1)^p - 1\right] \quad = 0 \text{ if } p \text{ is even and } -2 \text{ otherwise.}$$

$$\Rightarrow D_{2p+1} = -\frac{2aU_0}{(2p+1)\pi} \times \frac{2}{bSinh\left[\frac{(2p+1)\pi b}{a}\right]}$$

ie,

$$D_{2p+1} = -\frac{4aU_0}{(2p+1)b\pi Sinh\left[\frac{(2p+1)\pi b}{a}\right]}$$

Hence,

$$U(x,y) = -\sum_{p=0}^{\infty} \frac{4aU_0}{(2p+1)b\pi Sinh\left[\frac{(2p+1)\pi b}{a}\right]} Sin\frac{m\pi x}{a}Sinh\frac{m\pi y}{a}$$

3.2 Parabolic Differential Equation

The dimensionless temperature *T(x,t)* along the length of a rod of length l is known to be governed by the parabolic differential equation

$$\frac{\partial u}{\partial t} = \kappa \frac{\partial^2 u}{\partial x^2} \ , 0 \prec \pi x \prec l \ , t \succ 0$$

$$u(0,t) = 0, \ u(l,t) = 0, \ t \succ 0$$

$$u(x,0) = x^2 (l^2 - x^2), \ 0 \le x \le l$$

Solution

Using the same technique as in the previous examplewe assume $u(x,t) = X(x)T(t) \ne 0$

The differential equation then becomes

$$X(x)\frac{dT}{dt} = \kappa T(t)\frac{d^2X}{dx^2} \qquad (i)$$

On dividing through by $X(x)T(t)$ we obtain

$$\frac{1}{T(t)}\frac{dT}{dt} = \frac{\kappa}{X(x)}\frac{d^2X}{dx^2} \qquad (ii)$$

Using the same argument as with the elliptic equation this results into the following uncoupled ODEs:

$$\frac{d^2X}{dx^2} + m^2 X(x) = 0$$

$$(iii)$$

$$\frac{dT}{dt} + \lambda T(t), \qquad \lambda = \kappa m^2$$

with solutions

$$X(x) = ASinmx + BCosmx, \ T(t) = Pe^{-\lambda t}$$

From the boundary conditions

$$u(0,t) = X(0)T(t) = 0 \Rightarrow X(0) = 0 \qquad (iv)$$

Hence,

$$X(x) = ASinmx \qquad (v)$$

The second of the boundary conditions requires that

$$X\left(l\right)=0 \qquad (vi)$$

ie,

$$ASinml=0$$

$\because$ $X\left(x\right)\neq 0$ we require that

$$Sinml=0$$

ie,

$$ml=Sin^{-1}0=r\pi \qquad r\in\mathbb{Z}$$

Thus we have that

$$m_r=\frac{r\pi}{l}$$

Thatis,

$$X_r\left(x\right)=A_rSin\frac{r\pi x}{l} \qquad (vii)$$

Correspondingly we have

$$u_r\left(x,t\right)=X_r\left(x\right)T_r\left(t\right)=D_rSin\frac{r\pi x}{l}e^{-\lambda t}$$

Hence,

$$u\left(x,t\right)=\sum_{m=1}^{\infty}D_rSin\frac{r\pi x}{l}e^{-\lambda t}$$

The initial condition of the problem requires that

$$u\left(x,0\right)=\sum_{m=1}^{\infty}D_rSin\frac{r\pi x}{l}=x^2\left(l^2-x^2\right)$$

Thus,

$$\int_0^l\sum_{r=1}^{\infty}D_rSin\frac{r\pi x}{l}Sin\frac{p\pi x}{l}dx=\int_0^l x^2\left(l^2-x^2\right)Sin\frac{p\pi x}{l}dx$$

ie,

$$\sum_{r=0}^{\infty}D_r\int_0^l Sin\frac{r\pi x}{l}Sin\frac{p\pi x}{l}dx=\int_0^l x^2\left(l^2-x^2\right)Sin\frac{p\pi x}{l}dx$$

ie,

$$\frac{D_p}{2}\int_0^{\infty}\left(1-Cos\frac{2p\pi x}{l}\right)dx=\int_0^l x^2\left(l^2-x^2\right)Sin\frac{p\pi x}{l}dx$$

ie,

$$\frac{lD_p}{2} == \int_0^l x^2\left(l^2 - x^2\right)Sin\frac{p\pi x}{l}dx$$

RHS

$$\int_0^l x^2\left(l^2 - x^2\right)Sin\frac{p\pi x}{l}dx = \int_0^l \left(x^2l^2 - x^4\right)Sin\frac{p\pi x}{l}dx$$

$$= \left[-\frac{l}{p\pi}\left(x^2l^2 - x^4\right)Cos\frac{p\pi x}{l}\right]_0^l + \frac{l}{p\pi}\int_0^l\left(2xl^2 - 4x^3\right)Cos\frac{p\pi x}{l}dx$$

$$= \frac{l}{p\pi}\left[\frac{l}{p\pi}\left[\left(2xl^2 - 4x^3\right)Sin\frac{p\pi x}{l}\right]_0^l - \frac{l}{p\pi}\int_0^l\left(2l^2 - 12x^2\right)Sin\frac{p\pi x}{l}dx\right]$$

$$= -\left(\frac{l}{p\pi}\right)^2\left[-\frac{l}{p\pi}\left[\left(2l^2 - 12x^2\right)Cos\frac{p\pi x}{l}\right]_0^l - \frac{24l}{p\pi}\int_0^l xCos\frac{p\pi x}{l}dx\right]$$

$$= \left(\frac{l}{p\pi}\right)^3\left[10l^2\left(-1\right)^p - 2l^2\right] + 24\left(\frac{l}{p\pi}\right)^3\left[\frac{l}{p\pi}\left[xSin\frac{p\pi x}{l}\right]_0^l - \frac{l}{p\pi}\int_0^l Sin\frac{p\pi x}{l}dx\right]$$

$$= \left(\frac{l}{p\pi}\right)^3\left[10l^2\left(-1\right)^p - 2l^2\right] + \left(\frac{l}{p\pi}\right)^5\left[Cos\frac{p\pi x}{l}\right]_0^l$$

$$= \left(\frac{l}{p\pi}\right)^3\left[10l^2\left(-1\right)^p - 2l^2\right] + \left(\frac{l}{p\pi}\right)^5\left[\left(-1\right)^p - 1\right]$$

ie,

$$\frac{lD_p}{2} = \left(\frac{l}{p\pi}\right)^3\left[10l^2\left(-1\right)^p - 2l^2\right] + \left(\frac{l}{p\pi}\right)^5\left[\left(-1\right)^p - 1\right]$$

$\therefore$

$$D_p = \frac{2}{l}\left[\left(\frac{l}{p\pi}\right)^3\left[10l^2\left(-1\right)^p - 2l^2\right] + \left(\frac{l}{p\pi}\right)^5\left[\left(-1\right)^p - 1\right]\right] \qquad (viii)$$

Finally, we have

$$u(x,t) = \frac{2}{l}\sum_{p=1}^{\infty}\left[\left(\frac{l}{p\pi}\right)^3\left[10l^2\left(-1\right)^p - 2l^2\right] + \left(\frac{l}{p\pi}\right)^5\left[\left(-1\right)^p - 1\right]\right]Sin\frac{p\pi x}{l}e^{-\frac{\kappa p\pi}{l}t} \qquad (ix)$$

3.3 Hyperbolic Differential Equation.

The wave generated in a string of length ρ by a sudden impulse is known to satisfy the following Initial –Boundary Value Problem

$$\frac{\partial^2 U}{\partial t^2} = \beta^2 \frac{\partial^2 U}{\partial x^2}\ , \quad 0 \prec x \prec \rho, t \succ 0$$

$$U(0,t) = U(\rho,t) = 0,\ t \succ 0$$

$$U_t(x,0) = 0,\ U(x,0) = x(\rho - x),\ 0 \le x \le \rho$$

Determine the displacement profile $U(x,t)$.

Solution

As ususal we assume a separable solution $U(x,t) = X(x)T(t) \neq 0.$

Then the differential equation becomes

$$X(x)\frac{d^2T}{dt^2} = \beta^2 T(t)\frac{d^2X}{dx^2} \qquad (i)$$

ie,

$$\frac{1}{T(t)}\frac{d^2T}{dt^2} = \frac{\beta^2}{X(x)}\frac{d^2X}{dx^2} = -\xi^2 \qquad (\xi = \text{constant})$$

This results in the following pairs of Ordinary Differential Equation:

$$\frac{d^2T}{dt^2} + \xi^2 T(t) = 0 \qquad (ii)$$

$$\frac{d^2X}{dx^2} + \kappa^2 X(x) = 0 \qquad \left(\kappa = \frac{\xi}{\beta}\right) \qquad (iii)$$

Thus, we have

$$X(x) = ASin\kappa x + BCos\kappa x\ ,\ \ T(t) = PSin\beta t + QCos\beta t$$

Application of the boundary condition requires

$$X_m(x) = A_m Sin\frac{m\pi x}{\rho} \qquad (iv)$$

The initial condition $U_t\left(x,0\right)=0$ requires that $A=0$.. Hence we have

$$U_m\left(x,t\right)=D_m Sin\frac{m\pi x}{\rho}Cos\frac{m\pi\beta t}{\rho} \qquad (v)$$

Therefore the solution $U\left(x,t\right)$ of the initial - boundary value problem is given as

$$U\left(x,t\right)=\sum_{m=1}^{\infty}D_m Sin\frac{m\pi x}{\rho}Cos\frac{m\pi\beta t}{\rho} \qquad (vi)$$

Finally the second initial condition $U\left(x,0\right)=x\left(\rho-x\right)$ requires that

$$U\left(x,t\right)=\sum_{m=1}^{\infty}D_m Sin\frac{m\pi x}{\rho}=x\left(\rho-x\right) \qquad (vii)$$

Taking the Finite FourierSine transform of (vii) gives

$$\int_0^{\rho}\sum_{m=1}^{\infty}D_m Sin\frac{m\pi x}{\rho}Sin\frac{r\pi x}{\rho}dx=\int_0^{\rho}x\left(\rho-x\right)Sin\frac{r\pi x}{\rho}dx$$

ie,

$$\sum_{m=1}^{\infty}D_m\int_0^{\rho}Sin\frac{m\pi x}{\rho}Sin\frac{r\pi x}{\rho}dx=\int_0^{\rho}x\left(\rho-x\right)Sin\frac{r\pi x}{\rho}dx \qquad (viii)$$

ie,

$$\frac{D_r}{2}\int_0^{\rho}\left(1-Cos\frac{2m\pi x}{\rho}\right)dx=\int_0^{\rho}x\left(\rho-x\right)Sin\frac{r\pi x}{\rho}dx$$

ie,

$$\frac{\rho D_r}{2}=\int_0^{\rho}x\left(\rho-x\right)Sin\frac{r\pi x}{\rho}dx$$

RHS :

$$\int_0^{\rho}x\left(\rho-x\right)Sin\frac{r\pi x}{\rho}dx=\int_0^{\infty}\left(x\rho-x^2\right)Sin\frac{r\pi x}{\rho}dx$$

$$=\frac{\rho}{r\pi}\left[-\left(x\rho-x^2\right)Cos\frac{r\pi x}{\rho}\right]_0^{\rho}+\frac{\rho}{r\pi}\int_0^{\infty}\left(\rho-2x\right)Cos\frac{r\pi x}{\rho}dx$$

$$=\left(\frac{\rho}{r\pi}\right)^2\left\{\left[\left(\rho-2x\right)Sin\frac{r\pi x}{\rho}\right]_0^{\rho}-2\int_0^{\infty}Sin\frac{r\pi x}{\rho}dx\right\}$$

$$=2\left(\frac{\rho}{r\pi}\right)^3\left[Cos\frac{r\pi x}{\rho}\right]_0^{\rho}=2\left(\frac{\rho}{r\pi}\right)^3\left[\left(-1\right)^r-1\right] \qquad (ix)$$

Thus, $$\frac{\rho D_r}{2} = 2\left(\frac{\rho}{r\pi}\right)^3 \left[\left(-1\right)^r - 1\right]$$

$$\Rightarrow D_r = \frac{-8}{\rho}\left[\frac{\rho}{\left(2r+1\right)\pi}\right]^3 \qquad (x)$$

Hence, we finally have the result.

$$U\left(x,t\right) = \frac{-8}{\rho}\sum_{r=0}^{\infty}\left[\frac{\rho}{\left(2r+1\right)\pi}\right]^3 Sin\frac{m\pi x}{\rho} Cos\frac{m\pi\beta t}{\rho} \qquad (xi)$$

CHAPTER FOUR
APPLICATION OF DIFFERENTIAL EQUATION

Problem 1

An object with an initial temperature of 200° F is allowed to cool in a room whose temperature is 100° F. After 20minutes the temperature of the object was found to have dropped to 170° F. Determine the time for which the temperature of the object will be 140° F. What is the temperature of the body after 150minutes.

Solution

By Newton's law the rate of cooling is proportional to the difference in the temperature of the body and that of the environment.

Suppose the temperature of the cooling body be $T(t)$. Then the cooling process is governed by

$$\frac{dT}{dt} = \kappa\ (T - T_R) \qquad (4.1)$$

$T(0) = 200,\ T_R = 100.$

where κ is a constant of the cooling body and T_R is the room temperature.

From the defining differential equation

$$\frac{dT}{(T - T_R)} = -\kappa dt \qquad (4.2)$$

ie,

$$\int \frac{dT}{(T - T_R)} = -\int \kappa dt$$

ie,

$$\ln(T - T_R) = -\kappa t + \beta$$

ie,

$$T(t) = 100 + Ae^{-\kappa t}$$

From the initial conditions of the problem we have

$$T(0) = 200$$

$\Rightarrow \quad A = 100$

Hence the temperature of the body for all time i

$$T(t) = 100\left(1 + e^{-\kappa t}\right)$$

In order to determine the constant κ we solve the equation

$$\int_{200}^{170} \frac{dT}{(T - T_R)} = -\int_{0}^{20} \kappa dt = -20\kappa$$

$$\left[\ln\left(T - T_R\right)\right]_{200}^{170} = -20\kappa$$

ie,

$$\ln\left[\frac{7}{10}\right] = -20\kappa$$

$$\kappa = -\frac{1}{20}\ln\left[\frac{7}{10}\right] = 0.0178$$

To determine the time the cooling body will be at $140^\circ F$.

From above,

$$T(t) = 100\left(1 + e^{-\kappa t}\right)$$

ie,

$$e^{-\kappa t} - \frac{T(t)}{100} - 1 = \frac{T(t) - 100}{100}$$

ie,

$$e^{\kappa t} = \frac{100}{T(t) - 100}$$

$$t = \frac{1}{\kappa}\ln\left(\frac{100}{T(t) - 100}\right) = \frac{1}{0.0178}\ln\left(\frac{5}{2}\right) = 51.48 \min s$$

$$t = 51.48 \min s$$

Finally

$$T(150) = 100\left[1 + e^{-}\right] = 106.93^\circ F$$

Problem 2

At certain instance of time 250gm of a radioactive substance is present in a sample. After 10years it was observed that the sample contains210gm of the element. How much of the substance remain after 50years? Determine the half life of the element.

Sulution

The mathematical model governing the decay process is given as

$$\frac{dy}{dt} = -\alpha y(t) \qquad (\alpha = \text{constant}) \qquad (i)$$

ie,

$$\frac{dy}{dt} + \alpha y(t) = 0$$

Hence,

$$y(t) = Ae^{-\alpha t}$$

From the initial data we observe that $y(0) = 250$

Thus,

$$A = 250$$

giving

$$y(t) = 250e^{-\alpha t} \qquad (ii)$$

To determine the constant α we solve the equation

$$\int_{250}^{210} \frac{dy}{y(t)} = -\int_{0}^{10} \alpha dt = 10\alpha$$

ie,

$$\left[\ln y(t)\right]_{250}^{210} = -10\alpha$$

$$\alpha = -\frac{1}{10}\ln\left(\frac{21}{25}\right) = 0.0175435$$

But $y(t) = 250e^{-\alpha t}$

ie,

$$e^{\alpha t} = \frac{250}{y(t)}$$

$$\alpha t = \ln\left(\frac{250}{y(t)}\right)$$

$$\therefore \qquad t = \frac{1}{\alpha}\ln\left(\frac{250}{y(t)}\right)$$

Hence,

$$T_{1/2} = \frac{1}{\alpha}\ln\left(\frac{250}{125}\right) = \frac{1}{\alpha}\ln 2 = 39.7561$$

Problem3

A tank initially containing $100 litres$ of a solution that holds $40gm$ of a chemical. A solution containing $2gm/litre$ of a chemical runs into the tank at the rate of $2litres/\min$ and the mixture runs out at the rate of $3litres/\min$

How much of the chemical is in the tank after $80\min s$?

Solution

Let $x(t)$ be the quantity of chemical present in the mixture at time t .

Then the rate of change of chemical $\frac{dx}{dt}$ is the rate of inflow-rate of outflow.

The rate of inflow $= 4gm/\min$. $\qquad (i)$

At time t the volume of solution in the tank is;

$$(100-t)litres \qquad (ii)$$

The concentration of the solution at time t is therefore

$$\frac{x(t)}{(100-t)}gm/litre \qquad (iii)$$

The rate at which the chemical is leaving the tank is therefore

$$\frac{3x(t)}{(100-t)}gm/\min \qquad (iv)$$

Thge governing differential equation of the mixture is therefore

$$\frac{dx}{dt} = 4 - \frac{3x(t)}{(100-t)} \qquad (v)$$

Solution

Setting $z = 100-t$ $\qquad (vi)$

we have

$$\frac{dx}{dt} = \frac{dx}{dz}\frac{dz}{dt} = -\frac{dx}{dz} \qquad (vii)$$

The governing differential equation (v) therefore becomes

$$-\frac{dx}{dz} = 4 - \frac{3x}{z}$$

ie,

$$\frac{dx}{dz} = \frac{3x-4z}{z} \qquad (viii)$$

This is a homogeneous differential equation and so we assume

$$x = uz$$

giving

$$\frac{dx}{dz} = u + z\frac{du}{dz} = \frac{3uz-4z}{z} = 3u-4$$

ie,

$$z\frac{du}{dz} = 2(u-2)$$

ie,

$$\frac{du}{u-2} = 2\frac{dz}{z} \qquad (ix)$$

$\therefore$ $\ln(u-2) = \ln z^2 + \alpha$

$\Rightarrow$ $u = Az^2 + 2$

$\therefore$ $x(t) = \left(Az^2+2\right)z$

ie,

$\therefore$ $x(t) = \left[A(100-t)^2 + 2\right](100-t)$

At $t = 0,\ x(t) = 40$

$$\Rightarrow 100(10000A+2) = 40$$

ie,

$$10000A = -1.6$$

$\therefore$ $A = -0.00016$

Hence,

$$x(80) = 20[400A+2] = 38.72gm$$

Hence, after $80\min s$ there remains $38.72gm$ of the chemical in the mixture.

Problem 4

The initial population of a specie is 10000. After 15 days it increases to 12000. After a very long time the population stabilizes at 25,000. Derive an emphirical formula for the population model as a function of time. At what time will the population be $\frac{3}{2}$ times its initial value.

Solution

Let $y(t)$ be the population at any point in time. Then the model is governed by

$$\frac{dy}{dt} = \kappa y(\alpha - y) \qquad (i)$$

where

$\kappa =$ population increase parameter.

$\alpha =$ population saturation value.

In particular, α=25,000. The model is therefore governed by the differial equation

$$\frac{dy}{dt} = \kappa y(25000 - y) \qquad (ii)$$

ie,

$$\frac{dy}{y(25000-y)} = \kappa dt$$

ie,

$$\int \frac{dy}{y(25000-y)} = \int \kappa dt = \kappa t + \beta \qquad (iii)$$

Resolving $\frac{1}{y(25000-y)}$ into partial fraction we have

$$\frac{1}{y(25000-y)} = \frac{a}{y} + \frac{b}{(25000-y)} = \frac{a(25000-y)+by}{y(25000-y)} = \frac{(b-a)y + 25000a}{y(25000-y)}$$

$\Rightarrow$

$$(b-a)y + 25000a = 1$$

Thus

$$a = \frac{1}{25000} = b$$

Hence,

$$\frac{1}{y(25000-y)} = \frac{1}{25000}\left[\frac{1}{y} + \frac{1}{25000-y}\right]$$

$(4.4.3)$ therefore becomes

$$\frac{1}{25000}\int\left[\frac{1}{y} + \frac{1}{25000-y}\right]dy = \kappa t + \beta$$

ie

$$\int\left[\frac{1}{y} + \frac{1}{25000-y}\right]dy = 25000\kappa t + \varepsilon$$

$\Rightarrow \qquad \ln\left(\dfrac{y}{25000-y}\right)=25000\kappa t+\varepsilon$

But $y(0)=10000$

ie,

$$\varepsilon=\ln\left(\frac{2}{3}\right)=-0.4055 \qquad (iv)$$

Also we have that $y(15)=12000$

$\Rightarrow \qquad \left[\ln\left(\dfrac{y}{25000-y}\right)\right]_{10000}^{12000}=375000\kappa$

ie,

$$\ln\left[\frac{12}{13}\times\frac{15}{10}\right]=375000\kappa=0.3254$$

$\therefore \qquad \kappa=\dfrac{0.3254}{375000} \qquad (v)$

Recall that $\dfrac{y}{25000-y}=B\exp(25000\kappa t)$

Suppose $y(0)=y_0$

ie $\qquad B=\dfrac{y_0}{25000-y_0} \qquad (vi)$

Hence, $\dfrac{y}{25000-y}=\dfrac{y_0}{25000-y_0}\exp(25000\kappa t)$

ie, $\dfrac{25000-y}{y}=\dfrac{25000-y_0}{y_0}\exp(-25000\kappa t)$

$$\left(\frac{25000}{y}-1\right)=\left(\frac{25000}{y_0}-1\right)\exp(-25000\kappa t)$$

$$\left(1-\frac{25000}{y}\right)=\frac{(y_0-25000)}{y_0}\exp(-25000\kappa t)$$

$\Rightarrow$

$$\frac{25000}{y}=1-\left[\frac{(y_0-25000)}{y_0}\exp(-25000\kappa t)\right]=\left[\frac{y_0+(25000-y_0)\exp(-25000\kappa t)}{y_0}\right]$$

$$=1+\left(\frac{25000}{y_0}-1\right)\exp(-25000\kappa t)$$

Hence, finally we have as the emphirical formula for the population model as

$$y(t)=\frac{25000}{1+\left(\frac{25000}{y_0}-1\right)\exp(-25000\kappa t)} \qquad (vii)$$

To determine the time the population will be twice its original value. Recall that

$$y(t)=\frac{25000}{1+\left(\frac{25000}{y_0}-1\right)\exp(-25000\kappa t)}$$

ie,

$$y\left[1+\left(\frac{25000}{y_0}-1\right)\exp(-25000\kappa t)\right]=25000 \qquad (viii)$$

$\Rightarrow$

$$y\left[y_0+(25000-y_0)\exp(-25000\kappa t)\right]=25000y_0$$

$$\frac{25000y_0}{y}-y_0=(25000-y_0)\exp(-25000\kappa t)$$

$$\left(\frac{25000y_0}{y}-y_0\right)\exp(25000\kappa t)=25000-y_0$$

$$ie, \quad \exp(25000\kappa t)=\left[\frac{25000-y_0}{\frac{25000y_0}{y}-y_0}\right]=\left[\frac{\frac{25000}{y_0}-1}{\frac{25000}{y}-1}\right] \qquad (ix)$$

$$ie, \qquad 25000\kappa t=\ln\left[\frac{\frac{25000}{y_0}-1}{\frac{25000}{y}-1}\right]$$

$$t=\frac{1}{25000\kappa}\ln\left[\frac{\frac{25000}{y_0}-1}{\frac{25000}{y}-1}\right] \qquad (x)$$

$$=\frac{1}{25000\kappa}\ln\left(\frac{9}{4}\right)$$

$$=37.38yrs \qquad (xi)$$

CHAPTER 5
INTRODUCTION TO DIFFERENCE EQUATION

Consider the sequence $\{u_j\}$ of terms $u_{1,}u_{2,}u_{3,}u_{4,}..........,u_m$ on which the operator Δ is defined such that

$$\Delta u_p = u_{p+1} - u_p \ , p = 1,2,3,.........,m-1 \qquad 5.1$$

We observe therefore that (5.1) gives the following results:

$$\left.\begin{aligned} \Delta u_1 &= u_2 - u_1 \\ \Delta u_2 &= u_3 - u_2 \\ \Delta u_3 &= u_4 - u_3 \ \text{ etc} \end{aligned}\right\} 5.2$$

The expressions above are refered to as the first finite difference of the terms

Higher Order Finite Differences

In analogy to the order of a differential equation the following definitions are associated with the order of finite differences:

$$\begin{aligned} \Delta^2 u_p &= \Delta(\Delta u_p) = \Delta(u_{p+1} - u_p) \\ &= \Delta(u_{p+1}) - \Delta(u_p) \\ &= (u_{p+2} - u_{p+1}) - (u_{p+1} - u_p) \\ &= u_{p+2} - 2u_{p+1} + u_p \qquad 5.3 \end{aligned}$$

$$\begin{aligned} \Delta^3 u_p &= \Delta(\Delta^2 u_p) = \Delta(u_{p+2} - 2u_{p+1} + u_p) \\ &= \Delta(u_{p+2}) - 2\Delta(u_{p+1}) + \Delta u_p \\ &= (u_{p+3} - u_{p+2}) - 2(u_{p+2} - u_{p+1}) + (u_{p+1} - u_p) \\ &= u_{p+3} - 3u_{p+2} + 3u_{p+1} - u_p \qquad 5.4 \end{aligned}$$

In general therefore we have,

$$\Delta^m u_p = u_{p+m} - mu_{p+m-1} + \frac{m(m-1)}{2!} u_{p+m-2} + ... + (-1)^r \ {}^mC_r u_{p+m-r} + ... + (-1)^m u_p \quad 5.5$$

Definition

Any equation that establishes a functional relation between the finite differences $\Delta u_p, \Delta^2 u_p, \ \Delta^3 u_p, ..., \Delta^m u_p$ is referred to as a difference equation and the order of the equation being the other of the highest finite difference present in the equation.

Examples

$$\left.\begin{aligned}&\Delta u_p + 6^p u_p = 3p\\&\Delta^2 u_p - 15\Delta u_p + 6^p u_p = 0\\&\Delta^3 u_p + 5p\Delta^2 u_p - 15^p \Delta u_p + 6u_p = p^2\end{aligned}\right\}5.6$$

The difference equations in (5.6) are first, second and third order equations respectively. The second of the equation is homogeneous while the first and third are nonhomogeneous.

We observe that

$$\Delta u_p + 6^p u_p = 3p \neq \left(u_{p+1} - u_p\right) + 6^p u_p = 3p$$

$$= u_{p+1} + \left(6^p - 1\right)u_p = 3p$$

ie

$$u_{p+1} + \left(6^p - 1\right)u_p - 3p = 0$$

$$\Delta^2 u_p - 15\Delta u_p + 6^p u_p = 0 \quad \neq \left(u_{p+2} - 2u_{p+1} + u_p\right) - 15\left(u_{p+1} - u_p\right) + 6^p u_p = 0$$

ie

$$u_{p+2} - 17u_{p+1} + \left(16 + 6^p\right)u_p = 0$$

Finally,

$$\Delta^3 u_p + 5p\Delta^2 u_p - 15^p \Delta u_p + 6u_p = p^2 \neq \left(u_{p+3} - 3u_{p+2} + 3u_{p+1} - u_p\right) + 5p\left(u_{p+2} - 2u_{p+1} + u_p\right)$$

$$-15^p\left(u_{p+1} - u_p\right) = p^2$$

ie,

$$u_{p+3} + (5p - 3)u_{p+2} - \left(15^p - 3 + 10p\right)u_{p+1} + \left(15^p - 1 + 5p\right)u_p = p^2$$

5.2 Formulation of Difference Equations.

1 Consider the definition

$u_p = A4^p$ where A is a constant.

$\Rightarrow$

$$u_{p+1} = A4^{p+1} \qquad (i)$$

ie,

$$u_{p+1} = A4^{p+1} = 4A4^p = 4u_p$$

ie,

$$u_{p+1} - 4u_p = 0 \qquad (ii)$$

2 Given that $u_p = A + 6^p B$

Then

$$u_{p+1} = A + 6^{p+1}B \qquad (i)$$

from the first equation we have

$$B = \frac{u_p - A}{6^p} \qquad (ii)$$

Thus

$$u_{p+1} = A + 6^{p+1}\left(\frac{u_p - A}{6^p}\right) = A + 6\left(u_p - A\right)$$
$$= 6u_p - 5A \qquad (iii)$$

Also

$$u_{p+2} = A + 6^{p+2}B = A + 36\left(u_p - A\right)$$
$$=36u_p - 35A \qquad (iv)$$

Eliminating A from (iii) and (iv) we have

$$u_{p+2} - 7u_{p+1} + 6u_p = 0 \qquad (v)$$

Thus example 1 satisfies a first order difference equation while 2 satisfieas a second order difference equation. Indeeed, the general solution af an *mth* order difference equation contains *m* arbitrary constants just as in the case of differential equations..

5.3 Method of Solution of Difference Equations.

In our discussion of method of solution for difference equations we shall illustrate it with the following examples:

1 $u_{m+1} - \lambda_m u_m = 0,\ m \in \mathbb{N}$ $\left(\lambda_m \text{ is a known function of } m\right)$

Solution

We solve the equation using the following procedure:

$$u_2 = \lambda_1 u_1$$
$$u_3 = \lambda_2 u_2 = \lambda_1 \lambda_2 u_1$$
$$u_4 = \lambda_3 u_3 = \lambda_1 \lambda_2 \lambda_3 u_1$$
$$u_5 = \lambda_4 u_4 = \lambda_1 \lambda_2 \lambda_3 \lambda_4 u_1$$
....
$$u_p = \lambda_{p-1} u_{p-1} = \lambda_1 \lambda_2 \lambda_3 \lambda_4 \ldots\ldots\ldots\ldots \lambda_{p-3} \lambda_{p-2} \lambda_{p-1} u_1$$

Therefore, the general solution to the given difference equation is given as

$$u_r = u_1 \prod_{m=1}^{r-1} \lambda_m$$

We observe that the above equation is a homogeneous difference eqation, the
method of solution for nonhomogeneous equations is illustrated in the following example:

$$u_{m+1} - \lambda_m u_m = \beta_m$$

in which $\lambda_m$and β_m are given functions of *m.*

Solution

We assume a solution of the form

$$u_m = x_m + y_m \qquad (i)$$

in which x_m is a solution of the reduced equation

$$u_{m+1} - \lambda_m u_m = 0 \qquad (ii)$$

and y_m a solution of the nonhomogeneous equation

$$u_{m+1} - \lambda_m u_m = \beta_m \qquad (iii)$$

This procedure is indeed analogous to that adopted in the solution of nonhomogeneous ordinary differential equation.

Examples

1 Determine the solution of the first order difference equation

$$u_{p+1} + 7^p u_p = 0\ ,\ u_1 = 5\ ;\ \ m \geq 1$$

Solution

Rewrite the equation as

$$u_{p+1} = -7^p u_p$$

We thus have the following sequence:

$$u_2 = -7u_1$$
$$u_3 = -7^2 u_2 = 7\times 7^2 u_1$$
$$u_4 = -7^3 u_3 = -7\times 7^2 \times 7^3 u_1$$
$$u_5 = -7^4 u_4 = 7\times 7^2 \times 7^3 \times 7^4 u_1$$
$$u_6 = -7^5 u_5 = -7\times 7^2 \times 7^3 \times 7^4 \times 7^5 u_1$$

....

$$u_r = -7^{r-1} u_{r-1} = (-1)^{r-1}\, 7\times 7^2 \times 7^3 \times 7^4 \times 7^5 .... \times 7^{r-1} u_1$$

ie,

$$u_m = (-1)^{m-1}\, 7\times 7^2 \times 7^3 \times 7^4 \times 7^5 .... \times 7^{m-1} u_1 = 7^{m(m+1)} u_1$$

But $u_1 = 5$

Hence, the solution to the difference equation is

$$u_m = 5\times 7^{m(m+1)}$$

Therefore to solve the corresponding nonhomogeneous equation

$$u_{p+1} + 7^p u_p = p \qquad (i)$$

we assume the solution

$$u_p = x_p + y_p \qquad (ii)$$

where

$$x_{p+1} + 7^p x_p = 0 \qquad (iii)$$

and

$$y_{p+1} + 7^p y_p = p \qquad (iv)$$

Clearly,

$$x_m = 5\times 7^{m(m+1)} \qquad (v)$$

Suppose $y_p = A + Bp$

Then,

$$y_{p+1} = A + B(p+1)$$

ie,

$$A + B(p+1) + 7^p (A + Bp) = p$$

$$\Rightarrow \qquad \left[A + B + 7^p A\right] + \left[1 + 7^p\right] Bp = p$$

Comparing coefficients we have

$$\left(1+7^{p}\right)A+B=0$$

$$\left(1+7^{p}\right)B=1$$

Hence we have

$$B=\frac{1}{1+7^{p}} \ , A=-\frac{1}{\left(1+7^{p}\right)^{2}}$$

Thus,

$$y_{p}=\frac{p}{1+7^{p}}-\frac{1}{\left(1+7^{p}\right)^{2}}=\frac{p\left(1+7^{p}\right)-1}{\left(1+7^{p}\right)^{2}}$$

The general solution of the nonhomogeneous difference equation is therefore

$$u_{m}=5\times 7^{m(m+1)}+\frac{m\left(1+7^{m}\right)-1}{\left(1+7^{m}\right)^{2}}$$

2 Obtain the solution to the second order difference equation

$$u_{p+2}-8u_{p+1}+15u_{p}=0$$

Solution

Assume $u_{r}=x^{r}\neq 0$

Then

$$x^{r+2}-8x^{r+1}+15x^{r}=0$$

$\Rightarrow$ $\left(x^{2}-8x+15\right)x^{r}=0$

$\Rightarrow$ $x^{2}-8x+15=0$

ie,

$$(x-3)(x-5)=0$$

hence,

$$x=3,5$$

$\therefore$ $u_{m}=3^{m}A+5^{m}B$ (A and B are constants)

3 Prove that the difference equation

$$u_{p+2}-4u_{p+1}+5u_{p}$$

has a solution of the form

$$5^{p/2}\left[ACosp\vartheta+BSinp\vartheta\right]$$ (A and B are constants)

Solution

Assume the solution

$$u_r = x^r \neq 0$$

This gives the auxiliary equation

$$x^2 - 4x + 5 = 0$$

with roots $x_1 = 2 + i$ and $x_2 = 2 - i$

Expressing these in polar form we have*i*

$$x_1 = R\,e^{i\vartheta} \text{ and } x_2 = R\,e^{-i\vartheta}$$

where $R = \sqrt{5}$ and $\vartheta = Tan^{-1}\left(\frac{1}{2}\right)$

But $u_p = Ax_1{}^p + Bx_2{}^p$

$$= A\left(R\,e^{i\vartheta}\right)^p + B\left(R\,e^{-i\vartheta}\right)^p$$

$$= AR^p\,e^{i\vartheta p} + BR^p\,e^{-i\vartheta p}$$

$$= AR^p\left(Cos\vartheta + iSin\vartheta\right)^p + BR^p\left(Cos\vartheta - iSin\vartheta\right)^p$$

By De Movier's theorem this gives

$$AR^p\left(Cosp\vartheta + iSinp\vartheta\right) + BR^p\left(Cosp\vartheta - iSinp\vartheta\right)$$

$$= R^p\left[\left(A+B\right)Cosp\vartheta + i\left(A+B\right)Sinp\vartheta\right]$$

$$= 5^{p/2}\left(aCosp\vartheta + bSinp\vartheta\right)$$

Example 4

It is known that $u_0, u_1, u_2, u_3, \ldots\ldots\ldots\ldots\ldots\ldots$satisfy the recurrence relation

$$u_{p+2} - 2\alpha u_{p+1} + \left(\alpha^2 + \lambda^2\right)u_p = 0 \quad \left(p = 0,1,2,\ldots\ldots\ldots\right)\ u_0 = 0, u_1 = 1$$

prove that for $\lambda \succ 0$

$$u_m = \frac{1}{\lambda}\left(\alpha^2 + \lambda^2\right)^{1/2} Sin\left(mTan^{-1}\frac{\lambda}{\alpha}\right)$$

Solution

Assume $u_m = x^m$, then we have

$$\left[x^2 - 2\alpha x + \left(\alpha^2 + \lambda^2\right)\right]x^m = 0$$

ie,

$$x^2 - 2\alpha x + \left(\alpha^2 + \lambda^2\right) = 0$$

Thus,

$$x = \frac{2\alpha \pm \left[4\alpha^2 - 4\left(\alpha^2 + \lambda^2\right)\right]^{1/2}}{2} = \alpha \pm i\lambda$$

ie,

$$x_1 = \alpha + i\lambda, x_2 = \alpha - i\lambda,$$

$\Rightarrow$

$x_1 = R\left(Cos\vartheta + iSin\vartheta\right)$ and $x_2 = R\left(Cos\vartheta + -iSin\vartheta\right),\ R = \left(\alpha^2 + \lambda^2\right)^{1/2}\ ,\ \vartheta = Tan^{-1}\dfrac{\lambda}{\alpha}$

Thus,

$$u_m = A\left[R\left(Cos\vartheta + iSin\vartheta\right)\right]^m + B\left[R\left(Cos\vartheta - iSin\vartheta\right)\right]^m$$

$$=R^m\left\{A\left(Cosm\vartheta + iSinm\vartheta\right) + B\left(Cosm\vartheta - iSinm\vartheta\right)\right\}$$

$$= R^m\left(aCosm\vartheta + bSinm\vartheta\right)$$

Applying the auxiliary conditions we have

$$u_0 = a = 0$$

ie,

$$u_m = R^m bSinm\vartheta$$

$$u_1 = RbSin\vartheta = RbSin\left(Tan^{-1}\frac{\lambda}{\alpha}\right) = 1$$

$$b = R^{-1} = \left(\alpha^2 + \lambda^2\right)^{-1/2}$$

Exercise

Determine the solution to the initial value problem

$$u_{p+2} - 5u_{p+1} + 6u_p = 0\ ;\ u_1 = 1, u_2 = 3$$