EDO UNIVERSITY IYAMHO
MATHEMATICAL METHODS
DEPARTMENT OF MATHEMATICS /ICT

COURSE CODE MTH 211: MATHEMATICAL METHODS (3 UNITS)

Instructors:PROF. YOMI AIYESIMI, Alhassan Charity
email:yomi.aiyesimi@edouniversity.edu.ng
Lectures: Thursday, 8am – 10:00 am, LT1, phone: (+2348134809593)

Description: This course is intended to give the students a thorough knowledge of Mathematical Methods. This course covers advanced topics such asfunctions, maximum and minimum values of a function, functions of two several Independent variables,Jacobian, Dependent And Independent Functions, Method of Lagrange's Multiplier, Line and Multiple Integrals and Line Integral with respect to an Arc Length.

Prerequisites: Students should be familiar with the concepts of function theory, Rolle's theorem, Taylor's theorem, maximum and minimum values of a function,functions of two several Independent variables and have strong knowledge ofJacobian, Dependent And Independent Functions. Students should also be familiar with basic concepts of concept of Lagrange's Multiplier, Line and Multiple Integrals and Line Integral with respect to an Arc Length.

Assignments: We expect to have 6 individual homework assignments throughout the course in addition to a Mid-Term Test and a Final Exam. Home works are organized and structured as preparation for the midterm and final exam, and are meant to be a studying material for both exams. The goal of these projects is to have the students knowledge of mathematical methods.

Grading: We will assign 10% of this class grade to home works, 10% for class work and attendance,10% for the mid-term test and 70% for the final exam. The Final exam is comprehensive.

Textbook: The recommended textbook for this class are as stated:

ISBN: 058223803x
Title: *Mathematical Methods for Science Students*
Author: G. Stephenson,
Publisher: Longman Group UKltd Essex cm20 2je England
Lectures: Below is a description of the contents. We may change the order to accommodate the materials you need for the projects.

Introduction to Function Theory

Definition 1.1

An *infinitesimal* is a variable whose limit is zero. Hence a variable α is said to be an infinitesimal with respect to another variable β if the ratio

$$\frac{\alpha}{\beta} \to 0 \qquad (1.1)$$

Two *initesimals* α and β are said to be of the same order if $\exists$ positive constant κ so that

$$|\alpha| \le \kappa |\beta| \qquad (1.2)$$

This statement is expressed mathematically as

$$\alpha = O(\beta) \qquad (1.3)$$

For instance, $5x + x^2 = O(x)$ if $x \to 0$

but $5x + x^2 = O(x^2)$ if $x \to \infty$

Differentials

Definition 1.2

Let $y = f(x)$ be defined at the point x and its neighborhood. Then $f(x)$ is said to be differentiable if when x isgiven an arbitrarary increament δx then the corresponding increament δy of y is such that

$$\delta y = A\delta x + \varepsilon \delta x \qquad (1.4)$$

where A is independent of δx and $\varepsilon \to 0$ as $\delta x \to 0$

Then,

$$\frac{\delta y}{\delta x} = A + \varepsilon$$

$$\therefore \text{ as } \delta x \to 0 \ \frac{\delta y}{\delta x} \to \frac{dy}{dx} = f'(x) = A \qquad (1.5)$$

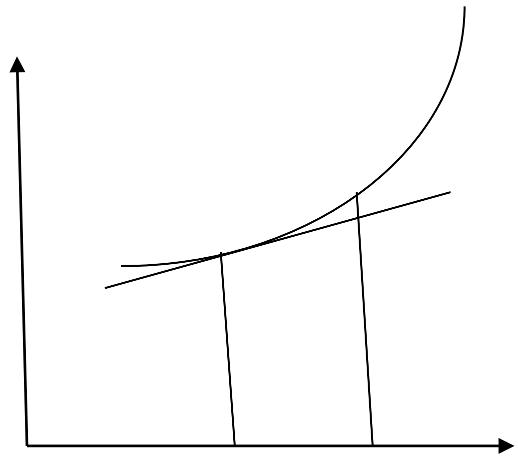
Observe that $dy = Adx = f'(x)dx$ (1.6)

In particular, if $y = f(x) = x \Rightarrow f'(x) = 1$

hence, $\delta y = 1.\ \delta x$ *ie*, $\delta y = \delta x$

That is, the differential of the dependent variable is equal to its increament.

The factor $f'(x)$ above is called the differential coefficient of y with respect to x. Thus, if $f(x)$ is differentiable then it has a finite derivative. This implies that $f(x)$ is differentiable at a point iff it has a finite derivative at that point.



Definition 1.3

$f(x)$ is differentiable in an interval $I = [a,b]$ if for any point $x \in I$ we have that

$$\frac{f(x+h)-f(x)}{h} \to \text{ a limit as } h \to 0 \text{ in any maner provided that } x+h \in I$$

Theorem 1.1

If $f(x)$ is differentiable at any point $x = x_0$ then $f(x)$ is continuous at $x = x_0$

Pr*oof*

At $x = x_0$ we have,

$$\delta y = A\delta x + \varepsilon \delta x; \quad \text{where } \varepsilon \to 0 \text{ as } \delta x \to 0, \ \delta y \to 0 \text{ as } \delta x \to 0$$

ie

$$f(x_0 + \delta x) \to f(x_0) \text{ as } \delta x \to 0$$

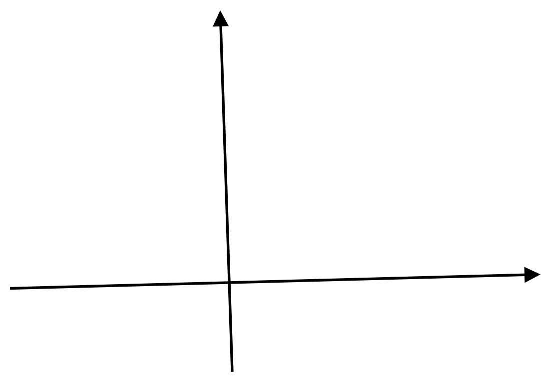
Also,

$$f(x) \to f(x_0) \text{ as } x \to x_0$$

$$\therefore \ f(x) \text{ is continuous at } x = x_0.$$

Theorem 1.2 (Rolle's Theorem)

If $f(x)$ is differentiable in (a,b) and continuous in $[a,b]$ and if $f(a) = f(b) = 0$, then $\exists$ a constant $\kappa \in [a,b]$ so that $f'(\kappa) = 0$



Pr *oof*

We have the following three possibilities:

(i) $f(x)=0$ in $[a,b] \Rightarrow f'(x)=0$

(ii) $f(x)\succ 0$ in $[a,b]$

(i) $f(x)\prec 0$ in $[a,b]$

The first case is a trivial case.

Suppose $f(x)\succ 0$ in $[a,b]$. Then it has a positive upper bound in $[a,b]$. Since it is continuous in $[a,b]$ this upper bound is at some point $\kappa \in (a,b)$ say. Hence, for any $h \succ 0$

$$\frac{f(\kappa+h)-f(\kappa)}{h} \le 0 \qquad (i)$$

and

$$\frac{f(\kappa-h)-f(\kappa)}{-h} \ge 0 \qquad (ii)$$

Since $f'(x)$ exists we recall that

$$\frac{f(\kappa+h)-f(\kappa)}{h} \to R\,f'(\kappa) = f'(\kappa) \;\; as \;\; h \to 0$$

and

$$\frac{f(\kappa-h)-f(\kappa)}{-h} \to Lf'(\kappa) = f'(\kappa) \;\; as \;\; h \to 0$$

The expressions in (i) and (ii) above implies that

$$f'(\kappa) \geq 0$$

and

$$f'(\kappa) \leq 0$$

But these two limits are the same by the uniqueness of the limit of a function

$$\therefore \quad f'(\kappa) = 0$$

Theorem 1.3 (First Mean Value Theorem)

If $f(x)$ is continuous in $[a,b]$ and differentiable in (a,b) then $\exists$ a constant $\kappa \in [a,b]$ so that

$$\frac{f(b)-f(a)}{b-a} = f'(\kappa)$$

Pr*oof*

Consider an auxiliary function

$$G(x) = f(x) - Ax$$

in which the parameter A is chosen so that

$$G(a) = G(b)$$

$\Rightarrow$

$$G(a) = f(a) - Aa \text{ and}$$
$$G(b) = f(b) - Ab$$

Thus, by definition we have that

$$f(a) - Aa = f(b) - Ab$$

$\Rightarrow$

$$A = \frac{f(b)-f(a)}{b-a} \qquad (i)$$

We observe that $G(x)$ satisfies the condition of Rolle's theorem in $[a,b]$, hence $\exists\ \kappa \in (a,b)$ so that $G'(\kappa) = 0$

ie,

$$f'(\kappa) - A = 0$$

ie,

$$A = f'(\kappa) \qquad (ii)$$

Comparing (i) and (ii) we finally have

$$\frac{f(b)-f(a)}{b-a} = f'(\kappa)$$

Corollary

$f(x)$ is constant in $[a,b]$ iff $f'(x)$=0 in $[a,b]$

Proof

If $f(x)$ *is* constant then $f'(x)$=0

Suppose $f'(x)=0$ in $[a,b]$ then $\exists\ \varepsilon \in [a,b]$ so that $f'(\varepsilon)$=0

ie,

$f'(x)=0$ in $[a,\varepsilon]$ and $f(x)$ satisfies the condition of Mean Value Theorem (MVT)

Therefore

$\exists\ \kappa \in [a,b]$ so that

$$\frac{f(\varepsilon)-f(a)}{\varepsilon-a} = f'(\kappa) = 0$$

$\Rightarrow$

$$f(\varepsilon) = f(a)$$

Now since ε is arbitrary $\Rightarrow$ $\forall\ x \in [a,b]\ f(x)$ is constant.

Theorem 1.4 $(Cauchy\ Theorem)$

If $f(x)$ and $g(x)$ are both continuous in $[a,b]$ and differentiable in (a,b) and if $g'(x) \neq 0$ in $[a,b]$ then $\exists$ a point $x=\kappa$ in (a,b) so that

$$\frac{f(b)-f(a)}{g(b)-g(a)} = \frac{f'(\kappa)}{g'(\kappa)}$$

Pr*oof*

Define an auxiliary function

$$G(x) = f(x) - Ag(x)$$

where A is chosen such that

$$G(a) = G(b)$$

ie,

$$f(b) - Ag(b) = f(a) - Ag(a)$$

ie,

$$f(b) - f(a) = Ag(b) - Ag(a) = A\big(g(b) - g(a)\big)$$

$\Rightarrow$

$$A = \frac{f(b)-f(a)}{g(b)-g(a)} \qquad (i)$$

We observe from the definition that $G(x)$ satisfies the condition of Rolle's theorem. Hence, $\exists\ \kappa \in [a,b]$ for which $G'(\kappa) = 0$

But from the definition of the auxiliary function $G(x)$ we have

$$G'(x) = f'(x) - Ag'(x)$$

and so by the satisfaction of the condition for Rolle's theorem we have that

$$f'(\kappa) - Ag'(\kappa) = 0$$

$\Rightarrow$

$$A = \frac{f'(\kappa)}{g'(\kappa)} \qquad (ii)$$

Hence, by virtue of (i) and (ii) we thus have

$$\frac{f(b) - f(a)}{g(b) - g(a)} = \frac{f'(\kappa)}{g'(\kappa)}$$

Note:

If $f(a) = g(a) = 0$ then the quotient $\frac{f(a)}{g(a)}$ is said to be indeterminate form.

Theorem 1.5

$f(a) = g(a) = 0$ and if $f'(x)$ exists in the neighborhood of $x = a$ and if $\frac{f'(x)}{g'(x)} \to L$ as $x \to a$ then $\frac{f(x)}{g(x)} \to L$ as $x \to a$. If $\frac{f'(x)}{g'(x)}$ is also indeterminate we repeat the process

ie,

$$\lim_{x \to a} \frac{f''(x)}{g''(x)} \to L \text{ if } \lim_{x \to a} \frac{f'(x)}{g'(x)} \to L \qquad \text{etc}$$

Pr*oof*

Let x be sufficiently close to a , then by Cauchy's theorem $\exists\ \kappa \in (a,b)$ so that

$$\frac{f(x) - f(a)}{g(x) - g(a)} = \frac{f'(\kappa)}{g'(\kappa)}$$

$\Rightarrow$

$$\frac{f'(\kappa)}{g'(\kappa)} = \frac{f(x)}{g(x)}$$

Theorem 1.6 (*Taylor's Theorem*)

If $f(x)$ and its first $(m-1)$ derivatives are continuous in $[a,b]$ and $f^{(m)}(x)$ exists in (a,b) then

$$f(b)=f(a)+(b-a)f'(a)+\frac{(b-a)^2}{2!}f''(a)+\frac{(b-a)^3}{3!}f'''(a)+........+\frac{(b-a)^{m-1}}{(m-1)!}f^{(m-1)}(a)+R_m$$

Proof

Consider the auxiliary function

$$F(x)=f(b)-f(x)-(b-x)f'(x)-\frac{(b-x)^2}{2!}f''(x)-\frac{(b-x)^3}{3!}f'''(x)-........-\frac{(b-x)^{m-1}}{(m-1)!}f^{(m-1)}(x)$$
$$-(b-x)^m\beta$$

Observe that $F(b)=0$

We chose β such that $F(a)=0$.

Hence, $F(x)$ satisfies the condition for Rolles's theorem $\Rightarrow\ \exists\ \kappa\ \in(a,b)$ so that $F'(\kappa)=0$

Observe that

$$F'(x)=-f'(x)+f'(x)-\left[(b-x)-2\frac{(b-x)}{2!}\right]f''(x)-\left[\frac{(b-x)^2}{2!}-3\frac{(b-x)^2}{3!}\right]f'''(x)$$
$$-\left[\frac{(b-x)^3}{3!}-4\frac{(b-x)^3}{4!}\right]f^{(iv)}(x)-............-\frac{(b-x)^{m-1}}{(m-1)!}f^{(m)}(x)+m(b-x)^{m-1}\beta \qquad (i)$$

But by definition

$$(b-a)^m\beta=f(b)-f(a)-(b-a)f'(a)-\frac{(b-a)^2}{2!}f''(a)-\frac{(b-a)^3}{3!}f'''(a)-$$
$$........-\frac{(b-a)^{m-1}}{(m-1)!}f^{(m-1)}(a) \qquad (ii)$$

Invoking Rolle's theorem in (i) we then have that

$$.-\frac{(b-\kappa)^{m-1}}{(m-1)!}f^{(m)}(x)+m(b-\kappa)^{m-1}\beta\ =0$$

ie,

$$\beta\ =\frac{(b-\kappa)^{m-r}}{m(m-1)!}f^{(m)}(x) \qquad (iii)$$

Substitution gives

$$f(b) = f(a) + (b-a) f'(a) + \frac{(b-a)^2}{2!} f''(a) + \frac{(b-a)^3}{3!} f'''(a) + \dots\dots.. + \frac{(b-a)^{m-1}}{(m-1)!} f^{(m-1)}(a) + R_m$$

where $R_m = \dfrac{(b-a)^m (b-\kappa)^{m-r}}{m(m-1)!} f^{(m)}(\kappa)$, $\kappa \in (a,b)$

R_m is called the remainder after m terms.

Thus we finally have that

$$f(a+h) = f(a) + hf'(a) + \frac{h^2}{2!} f''(a) + \frac{h^3}{3!} f'''(a) + \dots\dots.. + \frac{h^{m-1}}{(m-1)!} f^{(m-1)}(a) + R_m$$

Example

Compute the Taylor's series expansion of e^x about the origin.

Solution

$f(x) = e^x$.We recall that

$$f(x+a) = f(a) + xf'(a) + \frac{x^2}{2!} f''(a) + \frac{x^3}{3!} f'''(a) + \dots\dots.. + \frac{x^{m-1}}{(m-1)!} f^{(m-1)}(a) + R_m$$

At the origin $a = 0$

$$ie,\ f(x) = f(0) + x\,f'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(0) + \dots\dots.. + \frac{x^{m-1}}{(m-1)!} f^{(m-1)}(0) + R_m$$

Now given $f(x) = e^x$ we thus have

$$f(0) = 1,\ f'(0) = 1,\ f''(0) = 1,\ f'''(0) = 1$$

In general $f^{(p)}(0) = 1\ \forall\ p$

$$Thus, e^x = f(0) + x\,f'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(0) + \dots\dots.. + \frac{x^{m-1}}{(m-1)!} f^{(m-1)}(0) + R_m$$

$$= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots\dots\dots\dots\dots\dots\dots\dots.. + \frac{x^{m-1}}{(m-1)!} + R_m, \text{ where } R_m = \frac{x^m}{(m)!} e^x$$

Obseve that $\lim\limits_{m\to\infty} R_m = 0$

Exercise

(1) Compute the Maclaurin's series of the function $\ln\left(\dfrac{1+x}{1-x}\right)$

(2) Recalling that $e^{ix} = Cosx + i\ Sinx$, obtain the Taylor's series representation of the functions $Cosx$ and $Sinx$ about the origin.

MINIMUM AND MAXIMUM

*Definition*1.4

The point(s) $x = a$ where the derivative of the function $f(x)$ vanishes is (are) called the stationary (turning)point(s) of the function.

ie,

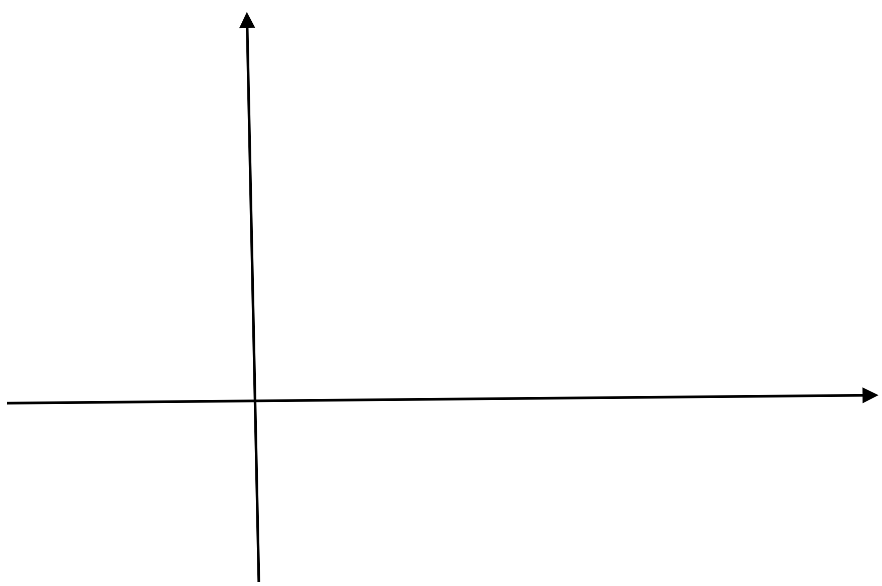
the turning points of a function $f(x)$ are determined by solving the algebraic equation

$$f'(x) = 0 \qquad (i)$$

Suppose $f(x)$ is defined in $[a,b]$ and a point $\kappa \in (a,b)$. If a sufficiently small quantity ε exists so that

$$\Delta f = f(x) - \varepsilon(\kappa)$$

keeps the same sign then we say that f has an extreme value at $x = \kappa$. This value is a $\max imum$ if $\Delta f \prec 0$ a minimum if $\Delta f \succ 0$



Theorem 1.7

If $f(x)$ has an extreme value at $x=\kappa$ then $f'(\kappa)=0$ provided that $f'(\kappa)$ exists.

Proof

Assume that $f(x)$ has a minimum at $x=\kappa$ then for sufficiently small quantity h we have

$$f(\kappa+h)-f(\kappa)\succ 0$$

and

$$f(\kappa-h)-f(\kappa)\succ 0$$

$\Rightarrow$

$$\frac{f(\kappa+h)-f(\kappa)}{h}\succ 0 \qquad (i)$$

and

$$\frac{f(\kappa-h)-f(\kappa)}{-h}\prec 0 \qquad (ii)$$

The limits of the above quotients as $h\to 0$ are respectively the right hand limit and the left hand limit. Since f is continuous at κ these two limits are equal

(i) gives that $f'(\kappa)\succ 0$ and (ii) gives that $f'(\kappa)\prec 0$

$$\therefore \quad f'(\kappa)=0$$

Note:

(1) $f(x)$ need not have an extreme value at a turning point (stationary point). Such a point is called a point of inf *lexion* (saddle point).

(2) $f(x)$ may have an etreme value at a point where $f'(x)\neq 0$. That is, $f'(\kappa)=0$ may not give all the extreme values of $f(x)$.

Examples

(i) Consider the function $f(x)=(x-1)^3$

$$f'(x)=0 \text{ at } x=1$$

The point $x=1$ is not one where $f(x)$ has an extreme value.

$(ii)\ f(x) = x^{2/3}$

(3) The greatest value of $f(x)$ in I need not be an extreme value.

Theorem 1.8

(1) If $f(x)$ is continuous in the neighborhood of $x = \kappa$

(2) $f'(\kappa) = f''(\kappa) = f'''(\kappa) = \ldots\ldots\ldots\ldots = f^{(m-1)}(\kappa) = 0,\ \ f^{(m)}(\kappa) \neq 0$

Then, $f(x)$ has no extreme value at $x = \kappa$ if m is odd, but if m is even then $f(x)$ has an extreme value at $x = \kappa$.

Maximum if $f^{(m)}(\kappa) \prec 0$

Minimum if $f^{(m)}(\kappa) \succ 0$

FUNCTIONS OF TWO (SEVERAL) INDEPENDENT VARIABLES.

Definition 2.1

The fubnction $f(x,y)$ of the independent variables x and y is said to be continuous at a point (x_0,y_0) if given $\varepsilon \succ 0 \ \exists \ \delta(\varepsilon,x,y)$ so that $|f(x,y)-f(x_0,y_0)| \prec \varepsilon$ when ever $|(x,y)-(x_0,y_0)| \prec \delta$

DIFFERENTIABILITY OF $f(x,y)$

Definition 2.2

Let $f(x,y)$ be a function of the two independent variables x and y and suppose δx and δy be the arbitrary increaments in x and y respectively giving rise to the corresponding increament δf in f. Then $f(x,y)$ is said to be differentiable at a point (x,y) if it has a definite value in the neighborhood of the point (x,y) and

$$\delta f = A\delta x + B\delta y + \varepsilon\rho \qquad (2.1)$$

where $\rho = |\delta x| + |\delta y|$, $\varepsilon \to 0$ as $\rho \to 0$, A, B are independent of δx and δy

(i) The partial derivative of f with respect to x is defined to be the derivative of f with respect to x holding y constant.

ie,
$$\lim_{\delta x \to 0} \frac{f(x+\delta x, y) - f(x,y)}{\delta x}$$

This derivative is denoted by $\frac{\partial f}{\partial x}$ or f_x

ie,
$$\frac{\partial f}{\partial x} = \lim_{\delta x \to 0}\left[\frac{f(x+\delta x, y) - f(x,y)}{\delta x}\right] \qquad (2.2)$$

(ii) In the same way, we define the partial derivative of f with respect to y as

$$\frac{\partial f}{\partial y} = \lim_{\delta y \to 0}\left[\frac{f(x, y+\delta y) - f(x,y)}{\delta y}\right] \qquad (2.3)$$

In gheneral, $\delta f = f(x+\delta x, y+\delta y) - f(x,y) \qquad (2.4)$

Putting $\delta y = 0$ in (2.4) we have

$$\delta f = f(x+\delta x, y) - f(x,y)$$

$$\Rightarrow \qquad \lim_{\delta x \to 0}\left(\frac{f(x+\delta x, y) - f(x,y)}{\delta x}\right) = A + \varepsilon$$

ie,

$$\delta f = A\delta x + \varepsilon|\delta x|, \text{ as } \delta x \to 0, \ \varepsilon \to 0$$

then,

$$\frac{\delta f}{\delta x} = A$$

Similarly,

$$\frac{\delta f}{\delta y} = B$$

hence,

$$\delta f = \frac{\delta f}{\delta x}\delta x + \frac{\delta f}{\delta y}\delta y + \varepsilon\rho \qquad (2.5)$$

The expression in (2.5) is defined as the total differential of f and is denoted by

$$df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy \qquad (2.6)$$

Theorem 2.1

If f has continuous partial derivative $f_x(x,y)$ and $f_y(x,y)$ at the point (x,y) then it is differentiable at the point (x,y).

Pr*oof*

$$\delta f = f(x+\delta x, y+\delta y) - f(x,y) \qquad (i)$$

$$= f(x+\delta x, y+\delta y) - f(x, y+\delta y) + f(x, y+\delta y) - f(x,y) \qquad (ii)$$

$$\frac{\partial}{\partial x} f(x, y+\delta y) = \lim_{\delta x \to 0} \frac{f(x+\delta x, y+\delta y) - f(x, y+\delta y)}{\delta x} \qquad (iii)$$

$$\frac{\partial f}{\partial y} = \frac{f(x, y+\delta y) - f(x,y)}{\delta y} \qquad (iv)$$

Using definition (iii) and (iv) in (ii) we have,

$$f(x+\delta x, y+\delta y) - f(x, y+\delta y) = \frac{\partial}{\partial x}\left[f(x, y+\delta y) + \varepsilon_1\right]\delta x \qquad (v)$$

and

$$f(x, y+\delta y) - f(x,y) = \frac{\partial}{\partial y}\left[f(x,y) + \varepsilon_2\right]\delta y \qquad (vi)$$

ε_1 and ε_2 are chosen in such a way that $\varepsilon_1 \to 0$ as $\delta x \to 0$ and $\varepsilon_2 \to 0$ as $\delta y \to 0$.

Since, $f_x(x,y)$ and $f_y(x,y)$ are continuous we may write

$$\frac{\partial}{\partial x}\left[f(x, y+\delta y)\right] \approx \left(\frac{\partial}{\partial x} f(x,y) + \varepsilon'\right)\delta x$$

ie,

equation (v) becomes

$$f(x+\delta x, y+\delta y) - f(x, y+\delta y) = \left[\frac{\partial}{\partial x} f(x,y) + \varepsilon_1 + \varepsilon' \right] \qquad (vii)$$

Substituting (vi) and (vii) into (ii) gives

$$df = \frac{\partial}{\partial x} f(x,y)\delta x + \frac{\partial}{\partial y} f(x,y)\delta y + (\varepsilon_1 + \varepsilon')\delta x + \varepsilon_2 \delta y \qquad (viii)$$

where both $(\varepsilon_1 + \varepsilon')$ and $\varepsilon_2 \to 0$ as δx and $\delta y \to 0$ respectively.

hence,

$$df = \frac{\partial f}{\partial x}\delta x + \frac{\partial f}{\partial y}\delta y$$

Thus, $f(x,y)$ is differentiable.

Theorem 2.2

If $f(x,y)$ is a differentiable function of x and y which are themselves differentiable functions of some independent variables s and t, then f considered as a function of s and t is differentiable.

Pr*oof*

By the differentiability of f with respect to x and y we have

$$df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy \qquad (1)$$

Also,
$$\delta x = \frac{\delta x}{\delta s}\delta s + \frac{\delta x}{\delta t}\delta t + \varepsilon'\rho \qquad (2)$$

$$\delta y = \frac{\delta y}{\delta s}\delta s + \frac{\delta y}{\delta t}\delta t + \varepsilon''\rho' \qquad (3)$$

Suppose $\rho = |\delta x| + |\delta y|$ and $\rho' = |\delta s| + |\delta t|$ where ε' and $\varepsilon'' \to 0$ as $\rho' \to 0$

Let $\omega = \max(\varepsilon', \varepsilon'')$ and $\mu = \max\left(\left|\frac{\delta x}{\delta s}\right|\left|\frac{\delta x}{\delta t}\right|, \left|\frac{\delta y}{\delta s}\right|\left|\frac{\delta y}{\delta t}\right|\right)$

then, $|\delta x| \prec (\omega + \mu)\rho'$ and $|\delta y| \prec (\omega + \mu)\rho'$

$$\therefore \qquad \frac{\rho}{\rho'} = \frac{|\delta x| + |\delta y|}{\rho'} \prec \frac{2(\omega + \mu)\rho'}{\rho'} = 2(\omega + \mu)$$

Since $2(\omega + \mu)$ is finite and independent of ρ', $\rho \to 0$ as $\rho' \to 0$

Since s and t are independent variables $\delta s = ds$ and $\delta t = dt$

Thus we have

$$\frac{\partial x}{\partial s}\delta s + \frac{\partial x}{\partial t}\delta t \to \frac{\partial x}{\partial s}ds + \frac{\partial x}{\partial t}dt = dx$$

and
$$\frac{\partial y}{\partial s}\delta s + \frac{\partial y}{\partial t}\delta t \to \frac{\partial y}{\partial s}ds + \frac{\partial y}{\partial t}dt = dy$$

But

$$\delta x = \frac{\delta x}{\delta s}\delta s + \frac{\delta x}{\delta t}\delta t + \varepsilon'\rho = dx + \varepsilon'\rho$$

and

$$\delta y = \frac{\delta y}{\delta s}\delta s + \frac{\delta y}{\delta t}\delta t + \varepsilon''\rho' = dy + \varepsilon''\rho'$$

$$\delta f = df + \varepsilon\rho \equiv \frac{\partial f}{\partial x}\delta x + \frac{\partial f}{\partial y}\delta y + \varepsilon\rho$$

$$= \frac{\partial f}{\partial x}\left(dx + \varepsilon'\rho'\right) + \frac{\partial f}{\partial y}\left(dy + \varepsilon''\rho'\right) + \varepsilon'\rho$$

$$= \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy + \rho'\left(\varepsilon'\frac{\partial f}{\partial x} + \varepsilon''\frac{\partial f}{\partial y} + \frac{\varepsilon'\rho}{\rho'}\right)$$

But the quantity

$$\left(\varepsilon'\frac{\partial f}{\partial x} + \varepsilon''\frac{\partial f}{\partial y} + \frac{\varepsilon'\rho}{\rho'}\right) \to 0 \text{ as } \rho \to 0$$

$$\rho' \to 0 \Rightarrow \varepsilon' \to 0, \varepsilon'' \to 0; \ \rho \to 0 \Rightarrow \varepsilon \to 0$$

$$\therefore \ df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy$$

If $f(x, y) = 0$ then y is defined as a function of x. In this case

$$df = 0 \Rightarrow \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy = 0$$

$$\frac{dy}{dx} = -\left(\frac{\partial f}{\partial x}\right)/\left(\frac{\partial f}{\partial y}\right) = -\left(\frac{\partial f}{\partial x}\right)\left(\frac{\partial f}{\partial y}\right)^{-1} \text{ provided } \frac{\partial f}{\partial y} \neq 0$$

Example

Given that $f(x, y) = x^3 + 3x^2y - 5xy^2 + 8y^3 = 0$. Determine $\frac{dy}{dx}$

Solution

$$f(x, y) = x^3 + 3x^2y - 5xy^2 + 8y^3 = 0$$

But $df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy = 0$

$$\frac{\partial f}{\partial y}dy = -\frac{\partial f}{\partial x}dx$$

ie, $\frac{dy}{dx} = -\left(\frac{\partial f}{\partial x}\right)\left(\frac{\partial f}{\partial y}\right)^{-1}$

Observe that

$$\frac{\partial f}{\partial x} = 3x^2 + 6xy - 5y^2$$

$$\frac{\partial f}{\partial y} = 3x^2 - 10xy + 24y^2$$

$$\therefore \qquad \frac{dy}{dx} = -\frac{3x^2 + 6xy - 5y^2}{3x^2 - 10xy + 24y^2}$$

Theorem 2.3

If $f(x,y)$ has continuous partial derivatives $f_x(x,y)$ and $f_y(x,y)$ then $f_{xy}(x,y) = f_{yx}(x,y)$ if they exist.

Pr*oof*

We shall prove this assertion using first principle.

Recall that

$$f_x(x,y) = \lim_{\delta x \to 0}\left[\frac{f(x+\delta x, y) - f(x,y)}{\delta x}\right] \text{ and}$$

$$f_y(x,y) = \lim_{\delta y \to 0}\left[\frac{f(x, y+\delta y) - f(x,y)}{\delta y}\right]$$

Now $f_{xy}(x,y) = \dfrac{\partial}{\partial y}\left(\dfrac{\partial f}{\partial x}\right)$

$$= \frac{\partial}{\partial y}\left(\lim_{\delta x \to 0}\left[\frac{f(x+\delta x, y) - f(x,y)}{\delta x}\right]\right)$$

$$= \lim_{\delta y \to 0}\left[\frac{f_x(x, y+\delta y) - f_x(x,y)}{\delta y}\right]$$

$$= \lim_{\delta y \to 0}\left[\frac{\lim_{\delta x \to 0}\left\{\frac{f(x+\delta x, y+\delta y) - f(x, y+\delta y)}{\delta x} - \frac{f(x+\delta x, y) - f(x,y)}{\delta x}\right\}}{\delta y}\right]$$

$$= \lim_{\delta y \to 0}\lim_{\delta x \to 0}\left[\frac{f(x+\delta x, y+\delta y) - f(x, y+\delta y) - f(x+\delta x, y) + f(x,y)}{\delta x \delta y}\right] \qquad (\Delta)$$

Similarly, $f_{yx}(x,y)=\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right)=\lim_{\delta x\to 0}\left[\frac{f_y(x+\delta x,y)-f_y(x,y)}{\delta x}\right]$

$$=\lim_{\delta x\to 0}\left[\frac{\lim_{\delta y\to 0}\left\{\frac{f(x+\delta x,y+\delta y)-f(x+\delta x,y)}{\delta y}-\frac{f(x,y+\delta y)-f(x,y)}{\delta y}\right\}}{\delta x}\right]$$

$$=\lim_{\delta x\to 0}\lim_{\delta y\to 0}\left[\frac{f(x+\delta x,y+\delta y)-f(x+\delta x,y)-f(x,y+\delta y)-f(x,y)}{\delta x\delta y}\right] \qquad (\Delta\Delta)$$

Comparing (Δ) and $(\Delta\Delta)$ we have

$$f_{xy}(x,y)=f_{yx}(x,y)$$

PARTIAL DERIVATIVES OF HIGHER ORDER

Suppose $f=f(x,y)$

then

$$df=\frac{\partial f}{\partial x}dx+\frac{\partial f}{\partial y}dy\Rightarrow d\equiv dx\frac{\partial}{\partial x}+dy\frac{\partial}{\partial y}$$

$$d^2f=d(df)=d\left(\frac{\partial f}{\partial x}dx+\frac{\partial f}{\partial y}dy\right)=d\left(\frac{\partial f}{\partial x}dx\right)+d\left(\frac{\partial f}{\partial y}dy\right)$$

$$=\left(dx\frac{\partial}{\partial x}+dy\frac{\partial}{\partial y}\right)\left(dx\frac{\partial f}{\partial x}\right)+\left(dx\frac{\partial}{\partial x}+dy\frac{\partial}{\partial y}\right)\left(dy\frac{\partial f}{\partial y}\right)=d^2x\frac{\partial^2 f}{\partial x^2}+dydx\frac{\partial^2 f}{\partial y\partial x}+dxdy\frac{\partial^2 f}{\partial x\partial y}+d^2y\frac{\partial^2 f}{\partial y^2}$$

$$=\left(d^2x\frac{\partial^2}{\partial x^2}+2dxdy\frac{\partial^2}{\partial x\partial y}+d^2y\frac{\partial^2}{\partial y^2}\right)f \quad =\left(dx\frac{\partial}{\partial x}+dy\frac{\partial}{\partial y}\right)^2 f$$

ie, $\qquad d^2f=\left(dx\frac{\partial}{\partial x}+dy\frac{\partial}{\partial y}\right)^2 f$

In general, $\quad d^mf=\left(dx\frac{\partial}{\partial x}+dy\frac{\partial}{\partial y}\right)^m f$

2JACOBIAN, DEPENDENT AND INDEPENDENT FUNCTIONS

*Definition*2.3

If u and v are two functions of x and y we say that they are *functionally dependent* if there is a functional relationship $f(u,v)=0$ or $u=\varphi(v)$. If there exists no such relationship we say that they are *functionally independent*. For instance, if $v=2x-3y$ and $u=4x^2-12xy+9y^2$ we observe that $u-v^2=0$, *ie*, $f(u,v)=0$. Therefore, the functions u and v are functionally dependent.

Suppose u and v are functionally dependent.

then

$$f(u,v)=0. \qquad (1)$$

Differentiating partially with respect to x we have

$$\frac{\partial f}{\partial u}\frac{\partial u}{\partial x}+\frac{\partial f}{\partial v}\frac{\partial v}{\partial x}=0 \qquad (2)$$

Differentiating partially with respect to y we have

$$\frac{\partial f}{\partial u}\frac{\partial u}{\partial y}+\frac{\partial f}{\partial v}\frac{\partial v}{\partial y}=0 \qquad (3)$$

Re-writting (2) and (3) in matrix-vector form we have

$$\begin{pmatrix}\frac{\partial u}{\partial x} & \frac{\partial v}{\partial x}\\ \frac{\partial u}{\partial y} & \frac{\partial v}{\partial y}\end{pmatrix}\begin{pmatrix}\frac{\partial f}{\partial u}\\ \frac{\partial f}{\partial v}\end{pmatrix}=\begin{pmatrix}0\\0\end{pmatrix} \qquad (4)$$

Equation (4) has a non-trivial solution if

$$\begin{vmatrix}\frac{\partial u}{\partial x} & \frac{\partial v}{\partial x}\\ \frac{\partial u}{\partial y} & \frac{\partial v}{\partial y}\end{vmatrix}=0 \text{ or } \begin{vmatrix}\frac{\partial u}{\partial x} & \frac{\partial u}{\partial y}\\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y}\end{vmatrix}=0 \qquad (5)$$

The determinant in (5) is refered to as the *Jacobian* of the functions u and v with respect to x and y denoted by

$$J=\frac{\partial(u,v)}{\partial(x,y)}$$

Example

$$u=\frac{x+y}{x}\text{ and }v=\frac{x-y}{y}$$

$$J(u,v)=\frac{\partial(u,v)}{\partial(x,y)}=\begin{vmatrix}\frac{\partial u}{\partial x} & \frac{\partial v}{\partial x}\\ \frac{\partial u}{\partial y} & \frac{\partial v}{\partial y}\end{vmatrix}=\begin{vmatrix}-\frac{y}{x^2} & \frac{1}{y}\\ \frac{1}{x} & -\frac{x}{y^2}\end{vmatrix}=\frac{xy}{x^2y^2}-\frac{1}{xy}=$$

ie, $$J(u,v)=\frac{1}{xy}-\frac{1}{xy}=0$$

Hence, the functions u and v are therefore functionally dependent.

In the opening example above, we have that $u=4x^2-12xy+9y^2$ and $v=2x-3y$

$$J(u,v)=\begin{vmatrix}8x-12y & 2\\ 18y-12x & -3\end{vmatrix}=3(12y-8x)-2(18y-12x)$$

ie, $$J(u,v)=36y-24x-36y+24x=0$$

showing the Jacobian to vanish.

Hence, a necessary condition for the functional dependence of any two functions is that their Jacobian must vanish.

RULES OF JACOBIAN

Let $u=u(x,y)$, $v=v(x,y)$, $x=x(s,t)$ and $y=y(s,t)$

$$\frac{\partial u}{\partial s}=\frac{\partial u}{\partial x}\frac{\partial x}{\partial s}+\frac{\partial u}{\partial y}\frac{\partial y}{\partial s},\quad \frac{\partial v}{\partial s}=\frac{\partial v}{\partial x}\frac{\partial x}{\partial s}+\frac{\partial v}{\partial y}\frac{\partial y}{\partial s}$$

$$\frac{\partial u}{\partial t}=\frac{\partial u}{\partial x}\frac{\partial x}{\partial t}+\frac{\partial u}{\partial y}\frac{\partial y}{\partial t},\quad \frac{\partial v}{\partial t}=\frac{\partial v}{\partial x}\frac{\partial x}{\partial t}+\frac{\partial v}{\partial y}\frac{\partial y}{\partial t}$$

$$\frac{\partial(u,v)}{\partial(x,y)}\times\frac{\partial(x,y)}{\partial(s,t)}=\begin{vmatrix}\frac{\partial u}{\partial x} & \frac{\partial u}{\partial y}\\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y}\end{vmatrix}\begin{vmatrix}\frac{\partial x}{\partial s} & \frac{\partial x}{\partial t}\\ \frac{\partial y}{\partial s} & \frac{\partial y}{\partial t}\end{vmatrix}$$

$$=\begin{vmatrix}\frac{\partial u}{\partial x}\frac{\partial x}{\partial s}+\frac{\partial u}{\partial y}\frac{\partial y}{\partial s} & \frac{\partial u}{\partial x}\frac{\partial x}{\partial t}+\frac{\partial u}{\partial y}\frac{\partial y}{\partial t}\\ \frac{\partial v}{\partial x}\frac{\partial x}{\partial s}+\frac{\partial v}{\partial y}\frac{\partial y}{\partial s} & \frac{\partial v}{\partial x}\frac{\partial x}{\partial t}+\frac{\partial v}{\partial y}\frac{\partial y}{\partial t}\end{vmatrix}=\begin{vmatrix}\frac{\partial u}{\partial s} & \frac{\partial u}{\partial t}\\ \frac{\partial v}{\partial s} & \frac{\partial v}{\partial t}\end{vmatrix}=\begin{vmatrix}\frac{\partial u}{\partial s} & \frac{\partial v}{\partial s}\\ \frac{\partial u}{\partial t} & \frac{\partial v}{\partial t}\end{vmatrix}=\frac{\partial(u,v)}{\partial(s,t)}$$

ie,

$$\frac{\partial(u,v)}{\partial(x,y)}\cdot\frac{\partial(x,y)}{\partial(s,t)}=\frac{\partial(u,v)}{\partial(s,t)}$$

METHOD OF LAGRANGE'S MULTIPLIER.

Given a function $f(x,y)$ whose stationary points we want to determine subject to a constraint equation $g(x,y)=0$

Then, we have $df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy = 0 \qquad (1)$

Equating the coefficients of dx and dy on both sides of vthe equation gives

$$\frac{\partial f}{\partial x} = 0, \quad \frac{\partial f}{\partial y} = 0 \qquad (2)$$

From the constraint equation we also have

$$dg = \frac{\partial g}{\partial x}dx + \frac{\partial g}{\partial y}dy = 0 \qquad (3)$$

Multiplying (3) by a parameter λ and adding the result to (1) we obtain

$$df + \lambda dg = \left(\frac{\partial f}{\partial x} + \lambda\frac{\partial g}{\partial x}\right)dx + \left(\frac{\partial f}{\partial y} + \lambda\frac{\partial g}{\partial y}\right)dy = 0 \qquad (4)$$

Choosing λ so that

$$\frac{\partial f}{\partial x} + \lambda\frac{\partial g}{\partial x} = 0 \qquad (5)$$

and $$\frac{\partial f}{\partial y} + \lambda\frac{\partial g}{\partial y} \qquad (6)$$

Thus, equation (5) & (6) together with the constraint equation $g(x,y)=0$ are sufficient to determine the stationary points and the Lagrages parameter λ.

Examples

(1) Determine the maximum distance from the origin $(0,0)$ to the curve $3x^2 + 3y^2 + 4xy - 2 = 0$

Solution

Distance from origin l^2 is given as

$$l^2 = f(x,y) = x^2 + y^2 \quad \left(ie,\ l = \sqrt{x^2 + y^2}\right)$$

$$g(x,y) = 3x^2 + 3y^2 + 4xy - 2 = 0$$

$$f_x = \frac{\partial f}{\partial x} = 2x\ ,\ f_y = \frac{\partial f}{\partial y} = 2y,\ \ g_x = \frac{\partial g}{\partial x} = 6x + 4y\ ,\ g_y = \frac{\partial g}{\partial y} = 6y + 4x$$

Thus,

$$2x+\lambda(6x+4y)=0$$
$$2y+\lambda(6y+4x)=0$$
$$3x^2+3y^2+4xy-2=0$$

ie,

$$4\lambda(x^2-y^2)=0 \quad \Rightarrow y=\pm x$$

If $y=-x,$ then

$$2x^2-2=0$$

ie,

$$x=\pm 1$$

If $y=x,$ then

$$10x^2-2=0$$

ie,

$$x=\pm\frac{1}{\sqrt{5}}$$

For $x=1$ we have,

$$2\lambda+2=0\Rightarrow \lambda=-1$$

The turning points of the functions are therefore,

$$A\left(-\frac{1}{\sqrt{5}},-\frac{1}{\sqrt{5}}\right),B\left(\frac{1}{\sqrt{5}},\frac{1}{\sqrt{5}}\right),C(1,-1) \text{ and } D(-1,1)$$

But $l^2=x^2+y^2 \Rightarrow l=\sqrt{x^2+y^2}$

$$l^2{}_A=\frac{2}{5},l^2{}_B=\frac{2}{5}$$

ie,

$$l_{A,B}=\sqrt{\frac{2}{5}}$$
$$l^2{}_C=2=l^2{}_D$$

hence,

$$l_{c,D}=\sqrt{2}$$

Therefore, maximum distance from the origin to the curve is attained at the points $C(1,-1)$ and $D(-1,1)$.

(2) Determine the extreme values (if any) of the function $f(x,y)=\left(x+\frac{1}{x}\right)^2+\left(y+\frac{1}{y}\right)^2$ along the straight line $x+y=1$

Solution

$$F(x,y)=f(x,y)+\lambda g(x,y) \qquad (a)$$

$$\frac{\partial F}{\partial x}=0 \Rightarrow \frac{\partial f}{\partial x}+\lambda\frac{\partial g}{\partial x}=0 \qquad (b)$$

$$\frac{\partial F}{\partial y}=0 \Rightarrow \frac{\partial f}{\partial y}+\lambda\frac{\partial g}{\partial y}=0 \qquad (c)$$

These result in the following equations:

$$2\left(x+\frac{1}{x}\right)\left(1-\frac{1}{x^2}\right)+\lambda=0 \qquad (d)$$

$$2\left(y+\frac{1}{y}\right)\left(1-\frac{1}{y^2}\right)+\lambda=0 \qquad (e)$$

$\Rightarrow$

$$\left(x-\frac{1}{x^3}\right)-\frac{\lambda}{2}=\left(y-\frac{1}{y^3}\right)$$

ie,

$$x-\frac{1}{x^3}=y-\frac{1}{y^3}$$

ie,

$$x^4y^3-y^3-x^3y^4+x^3=0$$

$$\Rightarrow \qquad x^3-y^3=x^3y^4-x^4y^3=x^3y^3(y-x)$$

ie,

$$(x-y)\left(x^2+xy+y^2\right)=-x^3y^3(x-y)$$

hence,

$$x^2+xy+y^2=-x^3y^3 \text{ or } x=y$$

But

$$x+y=1 \Rightarrow x=\frac{1}{2}=y$$

Recall that $f(x,y)=\left(x+\frac{1}{x}\right)^2+\left(y+\frac{1}{y}\right)^2 \Rightarrow f\left(\frac{1}{2},\frac{1}{2}\right)=2\left(2+\frac{1}{2}\right)^2=\frac{25}{2}$

Thus the function $f(x,y)=\left(x+\frac{1}{x}\right)^2+\left(y+\frac{1}{y}\right)^2$ has a minimum value at $\left(\frac{1}{2},\frac{1}{2}\right)$

Thus, the function $f(x,y)=\left(x+\frac{1}{x}\right)^2+\left(y+\frac{1}{y}\right)^2\geq\frac{25}{4}$, $x\succ 0, y\succ 0$

We may need to find the extreme values of a function $\omega = f(x,y,z,u)$ subject to the conditions;

$$\varphi(x,y,z,u)=0$$
$$\psi(x,y,z,u)=0$$

We follow the preceding method by introducing parameters λ and μ such that

$$F(x,y,z,u)=\omega+\lambda\varphi+\mu\psi$$

so that

$$\frac{\partial F}{\partial x}=\frac{\partial F}{\partial x}=\frac{\partial F}{\partial x}=\frac{\partial F}{\partial x}=0$$

together with $\varphi=0$ and $\psi=0$ and determine the point (x_0, y_0, z_0, u_0) where f may have an etreme value and determine the parameters λ and μ that produce this (these) extreme values.

Example

Find the extreme values of the function xyz on the surface of the positive octant of ellipsoid

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$$

Solution

$$\omega(x,y,z)=xyz,\quad \varphi(x,y,z)=\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$$

$$F(x,y,z)=xyz+\lambda\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}-1\right)=0 \qquad (i)$$

$$F_x=yz+\frac{2\lambda x}{a^2}=0 \qquad (ii)$$

$$F_y=xz+\frac{2\lambda y}{b^2}=0 \qquad (iii)$$

$$F_z=xy+\frac{2\lambda z}{c^2}=0 \qquad (iv)$$

together with $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$

Multiplying equation $(ii),(iii)$ and (iv) by x, y and z respectively and adding the result s yeild

$$3xyz+2\lambda\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}-1\right)=0$$

ie

$$3xyz+2\lambda=0$$

Also from equation $(ii),(iii)$ and (iv) we have that

$$\frac{x^2}{a^2} = -\frac{xyz}{2\lambda}, \quad \frac{y^2}{b^2} = -\frac{xyz}{2\lambda}, \quad \frac{z^2}{c^2} = -\frac{xyz}{2\lambda}$$

giving the results;

$$\frac{x^2}{a^2} = \frac{y^2}{b^2} = \frac{z^2}{c^2} = \frac{1}{3}$$

hence,

$$\frac{x}{a} = \frac{y}{b} = \frac{z}{c} = \sqrt{\frac{1}{3}} = \frac{1}{\sqrt{3}}, \quad x, y, z \succ 0$$

$$\therefore \qquad x = \frac{a}{\sqrt{3}}, y = \frac{b}{\sqrt{3}} \text{ and } z = \frac{c}{\sqrt{3}},$$

Hence, at the point $\left(\frac{a}{\sqrt{3}}, \frac{b}{\sqrt{3}}, \frac{c}{\sqrt{3}}\right)$ the function $\omega = f(x,y,z)$ is given as $\frac{abc}{3\sqrt{3}}$ on the positive $x-$axis.It is zero on both y and $z-$axes. Therefore the function has a maximum value at the point $\left(\frac{a}{\sqrt{3}}, \frac{b}{\sqrt{3}}, \frac{c}{\sqrt{3}}\right)$ given as $\frac{abc}{3\sqrt{3}}$.

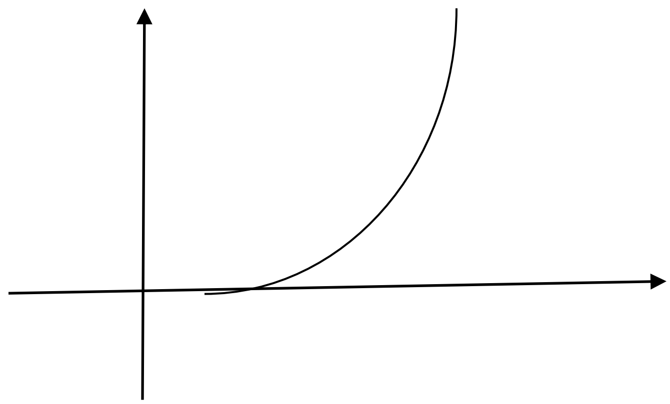
Exercise

Prove that $f(x,y) = x^3 + y^3 - 2\left(x^2 + y^2\right) + 3xy$ has stationary value at $(0,0)$ and $\left(\frac{1}{3}, \frac{1}{3}\right)$ and determine the nature of the stationary values.

LINE AND MULTIPLE INTEGRALS

3.1 LINE INTEGRALS

Suppose $y = f\left(x\right)$ is a real single-valued monotonic continuous function of x in $x_0 \prec x \prec x_f$



as represented in curve C with end points (x_0, y_0) and (x_f, y_f). If $P(x, y)$ and $Q(x, y)$ are two real single – valued continuous functions in x and y $\forall$ points on C then the integrals

$$\int_C P(x, y)\,dx \text{ and } \int_C Q(x, y)\,dy \qquad (3.1)$$

or the integral of the sum

$$\int_C \{P(x, y)\,dx + Q(x, y)\,dy\} \qquad (3.2)$$

are called the curvilinear or the line integral of the functions $P(x, y)$ and $Q(x, y)$ and the path of integration C being along $y = f(x)$ from A to B.

Now, since y is expressed in terms of the variable x each of the integrals in (3.1) and (3.2) is equivalent to

$$\int_C P(x, y)\,dx = \int_C P(x, f(x))\,dx \qquad (3.3)$$

$$\int_C Q(x, y)\,dy = \int_C Q(x, f(x))\,y'(x)\,dx \qquad (3.4)$$

hence,

$$\int_C \{P(x, y)\,dx + Q(x, y)\,dy\} = \int_C \{P(x, f(x))\,dx + Q(x, f(x))\,y'(x)\,dx\}$$

$$= \int_C \{P(x, f(x)) + Q(x, f(x))\,y'(x)\}\,dx$$

$$\int_C \{P(x,y)dx + Q(x,y)dy\} = \int_C \{P(x,f(x)) + Q(x,f(x))y'(x)\}dx \quad (3.5)$$

$$= \int_{x_0}^{x_f} \{P(x,f(x)) + Q(x,f(x))y'(x)\}dx$$

or $\Big\}$ (3.6)

$$\int_{y_0}^{y_f} \{P(g(y),y)g'(y) + Q(g(y),y)\}dy$$

Properties of Line Integrals

(1) $\int_a^b P(x,y)dx = \int_a^b P(x,f(x))dx = -\int_b^a P(x,f(x))dx$

(2) If the path of integration C is paralell to $y-$axis $ie,\ x=\kappa\ (\kappa\text{=constant})$ then,

$$\int_C P(x,y)dx = 0$$

Similarly, if C is paralell to $x-$axis $ie,\ y=\alpha\ (\alpha\text{=constant})$ $\int_C Q(x,y)dy = 0$

(3) If the path is divided into two paths by the point $D(x_p, y_p)$

ie, $x_0 \prec x_p \prec x_f$ and $y_0 \prec y_p \prec y_f$ then,

$$\int_C P(x,f(x))dx = \int_{x_0}^{x_f} P(x,f(x))dx = \int_{x_0}^{x_p} P(x,f(x))dx + \int_{x_p}^{x_f} P(x,f(x))dx$$

ie,

$$\int_A^B P(x,f(x))dx = \int_A^D P(x,f(x))dx + \int_D^B P(x,f(x))dx$$

and

$$\int_C P(g(y),y)dy = \int_{y_0}^{y_f} P(g(y),y)dy = \int_{y_0}^{y_p} P(g(y),y)dy + \int_{y_p}^{y_f} P(g(y),y)dy$$

ie,

$$\int_A^B P(g(y),y)dy = \int_A^D P(g(y),y)dy + \int_D^B P(g(y),y)dy$$

(4) If the path C is such that f is not single $-$ valued. Then

$$\int_C P(x,f(x))dx = \int_A^D P(x,y_1(x))dx + \int_D^B P(x,y_2(x))dx$$

where y_1 and y_2 are single $-$ valued in $x_0 \prec x \prec x_p$ and $x_p \prec x \prec x_f$ respectively.

*Example*3.1

Evaluate the integrals

(a) $\int_C (x+y)dx$ from $A(0,1)$ to $B(1,0)$ if

(i) $C=C_1$ is given as $y=1-x$

(ii) $C=C_2$ is given as $y=x^2$ from $O(0,0)$ to $P(1,1)$

(b) $\int_C \{(x^2+2y)dx+(x+y^2)dy\}$ from $A(0,1)$ to $R(2,3)$ if $C: \ y=1+x$

Solution

$(a(i))$ $\int_C (x+y)dx$ from $A(0,1)$ to $B(1,0)$ if $C=C_1$ is given as $y=1-x$

We observe that along the path of integration C, $y=1-x$

$\therefore \quad x+y=x+1-x=1$

hence,

$$\int_C (x+y)dx = \int_0^1 dx = [x]_0^1 = 1$$

(ii) Integrating along C_2 where $y=x^2$ from $O(0,0)$ to $P(1,1)$ we have

$$x+y=x+x^2 \ ,$$

hence,

$$\int_C (x+y)dx = \int_O^P (x+x^2)dx = \int_0^1 (x+x^2)dx = [x]_0^1 \left[\frac{x^2}{2}+\frac{x^3}{3}\right]_0^1 = \left(\frac{1}{2}+\frac{1}{3}\right) = \frac{5}{6}$$

(b) $\int_C \{(x^2+2y)dx+(x+y^2)dy\}$ from $A(0,1)$ to $R(2,3)$ if $C: \ y=1+x$

Observe that along the path of integration C, $y=1+x$

hence,

$$x^2+2y=2+2x+x^2 \text{ and } x+y^2 = x+(1+x)^2 = 1+3x+x^2,\ dy=dx$$

ie,

$$(x^2+2y)dx+(x+y^2)dy=(2+2x+x^2)dx+(1+3x+x^2)dx=(3+5x+2x^2)dx$$

ie,

$$\int_C \{(x^2+2y)dx+(x+y^2)dy\} \text{ from } A(0,1) \text{ to } R(2,3) \text{ if } C: \ y=1+x$$

$$= \int_0^2 (3+5x+2x^2)dx = \left[3x+\frac{5}{2}x^2+\frac{2}{3}x^3\right]_0^2 = 6+\frac{20}{2}+\frac{16}{3}=\frac{64}{3}$$

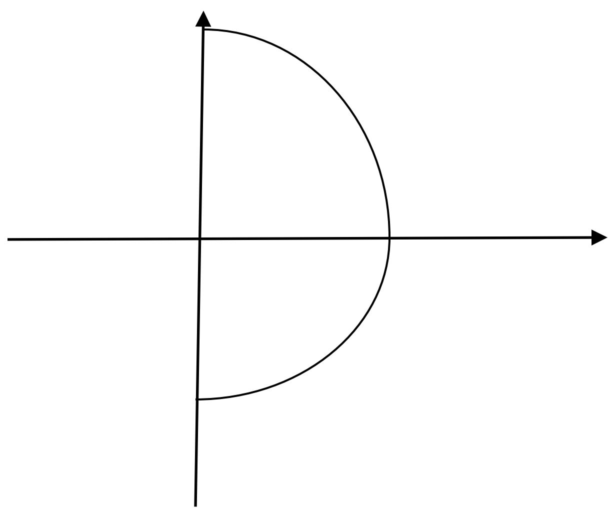
*Example*3.2

(a) Evaluate the line integral $\int_C (x+y)\,dx$ from $(0,1)$ to $(0,-1)$ where C is the semicircle defined as $C: y=\left(1-x^2\right)^{1/2}$.

Solution

We subdivide C into C_1 and C_2 thus;

$C_1 : y=\left(1-x^2\right)^{1/2}, C_2 : y=-\left(1-x^2\right)^{1/2}$



$$\therefore\ I=\int_A^D\left(x+\sqrt{1-x^2}\right)dx+\int_D^B\left(x-\sqrt{1-x^2}\right)dx$$

$$=\int_0^1\left(x+\sqrt{1-x^2}\right)dx+\int_1^0\left(x-\sqrt{1-x^2}\right)dx$$

$$=\int_0^1\left(x+\sqrt{1-x^2}\right)dx+\int_0^1\left(\sqrt{1-x^2}-x\right)dx$$

$$=\int_0^1\left(x+\sqrt{1-x^2}+\sqrt{1-x^2}-x\right)dx=\int_0^1 2\sqrt{1-x^2}\,dx$$

Putting $x = Sin\vartheta$

then,

$$\sqrt{1-x^2}=Cos\vartheta,\ dx=Cos\vartheta d\vartheta$$

hence,

$$I=2\int_0^{\pi/2}Cos^2\vartheta d\vartheta=\int_0^{\pi/2}\left(1+Cos2\vartheta\right)d\vartheta=\frac{\pi}{2}$$

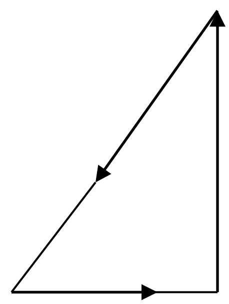
3.2 Line Integrals around Closed Plane Curves.

Example

Evaluate the line integral $\oint_C \left(2xydy - x^2 dx\right)$ where $C:\Delta$ whose vertices are at the points $(0,0),(1,0)$ and $(1,1)$.

Solution

We may sketch the Δ to make the problem clearer.



(i) Along $\overline{AB}$, $y = 0$

$\therefore \quad dy = 0$

hence,

$$2xydy - x^2dx = -x^2dx$$

$$\Rightarrow \quad \int_A^B \left(2xydy - x^2dx\right) = \int_A^B -x^2dx = -\int_0^1 x^2dx = -\left[\frac{x^3}{3}\right]_0^1 = -\frac{1}{3}$$

(ii) Along $\overline{BD}$, $x = 1$

$\therefore \quad dx = 0$

hence,

$$2xydy - x^2dx = 2ydy$$

$$\Rightarrow \quad \int_A^B \left(2xydy - x^2dx\right) = 2\int_A^B ydy = 2\int_0^1 ydx = \left[y^2\right]_0^1 = 1$$

Along $\overline{DA}$ both x and y are varying.

Since its a straight line we assume $y = ax + b$

$y(0) = b = 0 \Rightarrow y = ax,\ y(1) = 1 \Rightarrow y = x$ and $dy = dx$

$\therefore\ 2xydy - x^2dx = x^2dx$

$$\int_D^A \left(2xydy - x^2dx\right) = \int_1^0 x^2dx = -\int_0^1 x^2dx = -\left[\frac{x^3}{3}\right]_1^0 = -\frac{1}{3}$$

But $\oint_C \left\{P(x,y)dx + Q(x,y)dy\right\}$

$$= \int_A^B \left\{P(x,y)dx + Q(x,y)dy\right\} + \int_B^D \left\{P(x,y)dx + Q(x,y)dy\right\} + \int_D^A \left\{P(x,y)dx + Q(x,y)dy\right\}$$

ie,

$$\int_\Delta \left(2xydy - x^2dx\right) = \int_A^B \left(2xydy - x^2dx\right) + \int_B^D \left(2xydy - x^2dx\right) + \int_D^A \left(2xydy - x^2dx\right)$$

$$= -\frac{1}{3} + 1 - \frac{1}{3} = \frac{1}{3}$$

3.3 Line Integral with respect to an Arc Length.

To evaluate $\int P(x, y)ds$, where s is the length of an arc.

we recall that $s^2 = x^2 + y^2$

$$\Rightarrow \; s = \sqrt{x^2 + y^2}$$

ie,

$$\Delta s \cong \sqrt{(\Delta x)^2 + (\Delta y)^2}$$

$$\Rightarrow \; ds = \sqrt{d^2x + d^2y}$$

$$\therefore \; \frac{ds}{dx} = \sqrt{\left(\frac{dx}{dx}\right)^2 + \left(\frac{dy}{dx}\right)^2} = \sqrt{1 + y'^2}$$

hence,

$$ds = \sqrt{1 + y'^2}\,dx$$

On the otherhand if x and y are defined parametrically with respect to a parameter t we thus will have

$$ds = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} = \sqrt{\dot{x}^2 + \dot{y}^2}\,dt$$

Examples

(1) Evaluate the line integral $\oint_C (3x + 2xy)\,ds$ where C is defined as $C : y = x$ from $(0,0)$ to $(1,1)$.

Solution

Given that $C: y = x \Rightarrow dy = dx$ and $3x + 2xy = 3x + 2x^2$ on the curve C.

$ds = \sqrt{1 + y'^2}\,dx = \sqrt{2}dx$

$$\therefore \oint_C (3x + 2xy)ds = \int_0^1 (3x + 2x^2)\sqrt{2}dx = \sqrt{2}\int_0^1 (3x + 2x^2)dx = \sqrt{2}\left[\frac{3}{2}x^2 + \frac{2}{3}x^3\right]_0^1$$

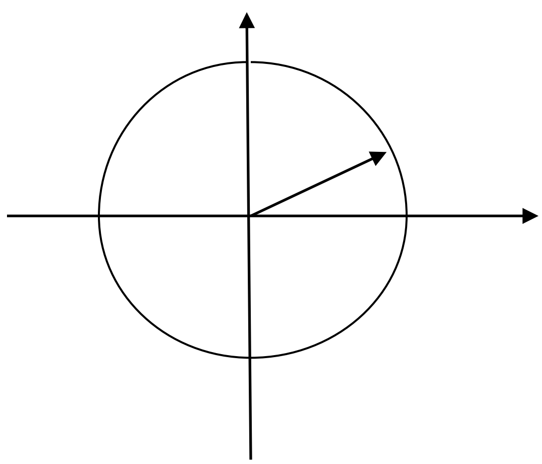
ie,

$$\oint_C (3x + 2xy)ds = \sqrt{2}\left(\frac{3}{2} + \frac{2}{3}\right) = \frac{13\sqrt{2}}{6}$$

(2) Evaluate $\oint_C (x^2 - y^2)ds$ where $C: x^2 + y^2 = 4$

Solution

The curve C is the circle centred $(0,0)$ with radius 2



In the segment ABC, $y = (4 - x^2)^{1/2}$

In the segment CDA, $y = -(4 - x^2)^{1/2}$

The coordinates of the points are: $A(2,0)$, $B(0,2)$, $C(-2,0)$ and $(0,-2)$

The integral $\oint_C (x^2 - y^2)ds$ will therefore be subdivided into 4 curvilinear integrals thus;

$$\oint_C (x^2 - y^2)ds = \int_A^B (x^2 - y^2)ds + \int_B^C (x^2 - y^2)ds + \int_C^D (x^2 - y^2)ds + \int_D^A (x^2 - y^2)ds$$

$$\text{Take } I_1 = \int_A^B \left(x^2 - y^2\right) ds; y = \left(4 - x^2\right)^{1/2} \therefore \ \frac{dy}{dx} = -x\left(4 - x^2\right)^{-1/2} = -\frac{x}{\sqrt{4 - x^2}}$$

$$\Rightarrow \int_A^B \left(x^2 - y^2\right) ds = \int_2^0 \left[x^2 - \left(4 - x^2\right)\right]\left(1 + \frac{x^2}{4 - x^2}\right)^{1/2} dx = \int_2^0 \left(2x^2 - 4\right)\left(1 + \frac{x^2}{4 - x^2}\right)^{1/2} dx$$

$$I_2 = \int_B^C \left(x^2 - y^2\right) ds; y = \left(4 - x^2\right)^{1/2} \therefore \ \frac{dy}{dx} = -x\left(4 - x^2\right)^{-1/2} = -\frac{x}{\sqrt{4 - x^2}}$$

$$\Rightarrow \int_A^B \left(x^2 - y^2\right) ds = \int_0^{-2} \left[x^2 - \left(4 - x^2\right)\right]\left(1 + \frac{x^2}{4 - x^2}\right)^{1/2} dx = \int_0^{-2} \left(2x^2 - 4\right)\left(1 + \frac{x^2}{4 - x^2}\right)^{1/2} dx$$

$$I_3 = \int_C^D \left(x^2 - y^2\right) ds; y = -\left(4 - x^2\right)^{1/2} \therefore \ \frac{dy}{dx} = x\left(4 - x^2\right)^{-1/2} = \frac{x}{\sqrt{4 - x^2}}$$

$$\Rightarrow \int_C^D \left(x^2 - y^2\right) ds = \int_{-2}^0 \left[x^2 - \left(4 - x^2\right)\right]\left(1 + \frac{x^2}{4 - x^2}\right)^{1/2} dx = \int_{-2}^0 \left(2x^2 - 4\right)\left(1 + \frac{x^2}{4 - x^2}\right)^{1/2} dx$$

$$I_4 = \int_D^A \left(x^2 - y^2\right) ds; y = \left(4 - x^2\right)^{1/2} \therefore \ \frac{dy}{dx} = -x\left(4 - x^2\right)^{-1/2} = -\frac{x}{\sqrt{4 - x^2}}$$

$$\Rightarrow \int_D^A \left(x^2 - y^2\right) ds = \int_0^2 \left[x^2 - \left(4 - x^2\right)\right]\left(1 + \frac{x^2}{4 - x^2}\right)^{1/2} dx = \int_0^2 \left(2x^2 - 4\right)\left(1 + \frac{x^2}{4 - x^2}\right)^{1/2} dx$$

$$\text{But } \oint_C \left(x^2 - y^2\right) ds = \int_A^B \left(x^2 - y^2\right) ds + \int_B^C \left(x^2 - y^2\right) ds + \int_C^D \left(x^2 - y^2\right) ds + \int_D^A \left(x^2 - y^2\right) ds$$

$$= I_1 + I_2 + I_3 + I_4$$

$$= \int_2^0 \left(2x^2 - 4\right)\left(1 + \frac{x^2}{4 - x^2}\right)^{1/2} dx + \int_0^{-2} \left(2x^2 - 4\right)\left(1 + \frac{x^2}{4 - x^2}\right)^{1/2} dx + \int_{-2}^0 \left(2x^2 - 4\right)\left(1 + \frac{x^2}{4 - x^2}\right)^{1/2} dx$$

$$+ \int_0^2 \left(2x^2 - 4\right)\left(1 + \frac{x^2}{4 - x^2}\right)^{1/2} dx$$

$$= \int_0^2 \left(2x^2 - 4\right)\left[\left(1 + \frac{x^2}{4 - x^2}\right)^{1/2} - \left(1 + \frac{x^2}{4 - x^2}\right)^{1/2}\right] dx + \int_0^{-2} \left(2x^2 - 4\right)\left[\left(1 + \frac{x^2}{4 - x^2}\right)^{1/2} - \left(1 + \frac{x^2}{4 - x^2}\right)^{1/2}\right] dx = 0$$

3.4 Line Integral Independent of Path.

Consider the integral

$$I = \int_A^B dF \qquad (3.4.1)$$

$$= \int_A^B \frac{dF}{dt} dt = \int_A^B \left(\frac{\partial F}{\partial x}\frac{dx}{dt} + \frac{\partial F}{\partial y}\frac{dy}{dt} \right) dt$$

$$= \int_A^B \left(\frac{\partial F}{\partial x} dx + \frac{\partial F}{\partial y} dy \right) = F(t_B) - F(t_A)$$

$$= F(x_B, y_B) - F(x_A, y_A) \qquad (3.4.2)$$

This integral depends only on the value of the function $F(x, y)$ at the end points A and B and not on the equation of the curve C.

If therefore C is a closed - simple curve then the points A and B coincide and so the value of the integral is zero.

ie, $$\oint_C dF = 0 \qquad (3.4.3)$$

Hence, if a function $F(x, y)$ can be found so that

$$\frac{\partial F}{\partial x} = P(x, y) \text{ and } \frac{\partial F}{\partial y} = Q(x, y), \text{ then}$$

$$\int_A^B \left(P(x, y)\ dx + Q(x, y)dy\right) = F_B - F_A \qquad (3.4.4)$$

The condition for the integral $I = \int_A^B \left(P(x, y)\ dx + Q(x, y)dy\right)$ to be independent of the path of integration C is therefore that $P(x, y) = \frac{\partial F}{\partial x}$ and $Q(x, y) = \frac{\partial F}{\partial y}$

But $$\frac{\partial^2 F}{\partial x \partial y} = \frac{\partial^2 F}{\partial y \partial x}$$

$$\Rightarrow \quad \frac{\partial}{\partial x}\left(\frac{\partial F}{\partial y}\right) = \frac{\partial Q(x, y)}{\partial x} = \frac{\partial}{\partial y}\left(\frac{\partial F}{\partial x}\right) = \frac{\partial P(x, y)}{\partial y}$$

ie, $$\frac{\partial Q(x, y)}{\partial x} = \frac{\partial P(x, y)}{\partial y} \qquad (3.4.5)$$

Therefore, the necessary condition for the integral in $(3.4.4)$ to be independent of path is equation $(3.4.5)$.

Examples

(1) Compute the line integral $\int_A^B (yCosxdx + \sin xdy)$ from the point $A(0,0)$ to $B\left(\frac{\pi}{4}, \frac{\pi}{4}\right)$.

Solution

Comparing the integral $\int_A^B (yCosxdx + Sinxdy)$ with the general form $\int_A^B (P(x,y)dx + Q(x,y)dy)$ we observe that

$$P(x,y) = yCosx \text{ and } Q(x,y) = Sinx$$

For exactness we require that

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$$

But from the given problem

$$\frac{\partial P}{\partial y} = Cosx \text{ and } \frac{\partial Q}{\partial x} = Cosx$$

ie,

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$$

and hence, the integrand is an exact differential. Thus, $\exists$ a function $F(x,y)$ such that

$$\frac{\partial F}{\partial x} = P(x,y) = yCosx \text{ and } \frac{\partial F}{\partial y} = Q(x,y) = Sinx$$

in which

$$\int_A^B (yCosxdx + Sinxdy) = \int_A^B dF$$

ie,

$$F(x,y) = \int_A^B yCosxdx = ySinx + h(y)$$

But $\frac{\partial F}{\partial y} = Sinx \Rightarrow \frac{\partial}{\partial y}\left[ySinx + h(y)\right] = Sinx + h'(y) = Sinx \Rightarrow h(y) = \beta$

hence,

$$F(x,y) = ySinx$$

$$\therefore \int_A^B (yCosxdx + Sinxdy) = [ySinx]_A^B = \frac{\pi}{4} Sin\frac{\pi}{4} = \frac{\pi}{4}\frac{1}{\sqrt{2}} = \frac{\pi}{4\sqrt{2}}$$

(2) Compute the integral $\oint_C \frac{xdy - ydx}{x^2 + y^2}$

Solution

Observe that

$$\frac{xdy - ydx}{x^2 + y^2} = \frac{x}{x^2 + y^2}dy - \frac{y}{x^2 + y^2}dx$$

ie,

$$P(x, y) = -\frac{y}{x^2 + y^2} \text{ and } Q(x, y) = \frac{x}{x^2 + y^2}$$

$$\frac{\partial P}{\partial y} = \frac{\partial}{\partial y}\left[-\frac{y}{x^2 + y^2}\right] = -\frac{\partial}{\partial y}\left[\frac{y}{x^2 + y^2}\right] = -\left[\frac{x^2 + y^2 - 2y^2}{\left(x^2 + y^2\right)^2}\right] = \frac{y^2 - x^2}{\left(x^2 + y^2\right)^2}$$

$$\frac{\partial Q}{\partial x} = \frac{\partial}{\partial x}\left[\frac{x}{x^2 + y^2}\right] = \left[\frac{x^2 + y^2 - 2x^2}{\left(x^2 + y^2\right)^2}\right] = \frac{y^2 - x^2}{\left(x^2 + y^2\right)^2}$$

$$\Rightarrow \frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$$

Thus, the integrand $\frac{xdy - ydx}{x^2 + y^2}$ is an exact differential of some function $F(x, y)$ such that

$$\frac{\partial F}{\partial x} = -\frac{y}{x^2 + y^2} \text{ and } \frac{\partial F}{\partial y} = \frac{x}{x^2 + y^2}$$

ie,

$$dF = \frac{x}{x^2 + y^2}dy - \frac{y}{x^2 + y^2}dx$$

hence,

$$\oint_C \frac{xdy - ydx}{x^2 + y^2} = \oint_C dF = 0$$

3.5 Multiple Integrals.

The integral

$$\int_{y=\alpha}^{y=\beta}\int_{x=a}^{x=b} f(x,y)\,dxdy \qquad (3.5.1)$$

is refered to as the multiple integral of the function $f(x,y)$ in the rectangular region

$$\Re: a \le x \le b;\ \alpha \le y \le \beta$$

In general, if $f(x,y)$ is continuous in $\Re$ then the order of integration does not affect the value of the integral.

ie, provided f is continuous in $\Re$

$$\iint_{\Re} f(x,y)\,dxdy = \iint_{\Re} f(x,y)\,dydx \qquad (3.5.2)$$

Properties of Double Integrals

For any continuous single-valued fuctions $f(x,y)$ and $g(x,y)$ in $\Re$

(a) $\iint_{\Re}\{f(x,y)+g(x,y)\}\,dxdy = \iint_{\Re} f(x,y)\,dxdy + \iint_{\Re} g(x,y)\,dxdy$

(b) For any constant α $\iint_{\Re} \alpha f(x,y)\,dxdy = \alpha \iint_{\Re} f(x,y)\,dxdy$

(c) If $\Re$ can be subdivided into two non-overlapping regions $\Re_1$ and $\Re_2$ $(\Re_1 \cup \Re_2 = \Re, \Re_1 \cap \Re_2 \varnothing)$ *then,*

$$\iint_{\Re} f(x,y)\,dxdy = \iint_{\Re_1} f(x,y)\,dxdy + \iint_{\Re_2} f(x,y)\,dxdy$$

Examples

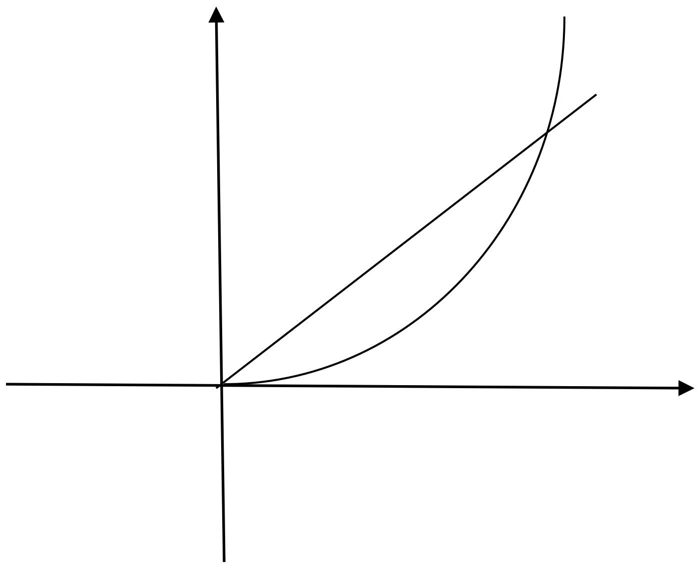
(1) Compute the double integral $\iint_{\Re}(2x^2+y)\,dxdy$ where $\Re$ is the region bounded by the straight line $y = x$ and the parabola $y = x^2$.

Solution

We neeed to obtain the boundary of the region $\Re$. This is effected by solving simultaneously the equations;

$$y = x \text{ and } y = x^2.$$

The solution is given by $x = 0,1$. That is, the straight line $y = x$ and the parabola $y = x^2$ meet at the points $(0,0)$ and $(1,1)$.



$$\therefore \quad I = \int_0^1 dy \int_{x=y}^{x=\sqrt{y}} \left(2x^2 + y\right) dx = \int_0^1 \left[\frac{2}{3}x^3 + xy\right]_y^{\sqrt{y}} dy$$

$$= \int_0^1 \left(\frac{5}{3}y^{\frac{3}{2}} - \frac{2}{3}y^3 - y^2\right) dy = \left[\frac{2}{3}y^{\frac{5}{2}} - \frac{1}{6}y^4 - \frac{1}{3}y^3\right]_0^1$$

$$= \left(\frac{2}{3} - \frac{1}{6} - \frac{1}{3}\right) = \frac{1}{6}$$

It can easily be shown using the same procedure above that $\int_{x=y}^{x=\sqrt{y}} dx \int_0^1 \left(2x^2 + y\right) dy$ will yeild the same result.

(2) Obtain the value of the integral $\int_0^1 \int_0^1 \frac{x-y}{(x+y)^3} dxdy$

Solution

In this problem since the function $f(x,y) = \frac{x-y}{(x+y)^3}$ has discontinuity along the straight line $y = -x$ contained in the rectangular region $\Re : 0 \le x \le 1;\ 0 \le y \le 1$. The order of integration therefore affects the value of the integral as will be shown presently.

Suppose $I_1 = \int_0^1 \int_0^1 \frac{x-y}{(x+y)^3} dxdy = \int_0^1 dy \int_0^1 \frac{x-y}{(x+y)^3} dx$

Setting $x + y = u \ \ ie, \ x = u - y \Rightarrow dx = du$

ie,

$$I_1 = \int_0^1 dy \int_0^1 \frac{x-y}{(x+y)^3} dx \equiv \int_0^1 dy \int_y^{1+y} \frac{u-2y}{u^3} du \equiv \int_0^1 dy \int_y^{1+y} (u-2y)u^{-3} du = \int_0^1 dy \int_y^{1+y} \left(u^{-2} - 2u^{-3}y\right) du$$

$$= \int_0^1 \left[\frac{y}{u^2} - \frac{1}{u}\right]_y^{1+y} dy = -\int_0^1 \frac{dy}{(1+y)^2}$$

Assume $1+y=\vartheta$, then $dy = d\vartheta$

ie,

$$I_1 = -\int_1^2 \frac{d\vartheta}{\vartheta^2} = \left[\frac{1}{\vartheta}\right]_1^2 = \left(\frac{1}{2}-1\right) = -\frac{1}{2}$$

On the other hand suppose $I_2 = \int_0^1 \int_0^1 \frac{x-y}{(x+y)^3} dydx = \int_0^1 dx \int_0^1 \frac{x-y}{(x+y)^3} dy$

Set $x+y=t$ so that $y = t-x$ and $dy = -dt$

ie,

$$I_2 = \int_0^1 dy \int_0^1 \frac{x-y}{(x+y)^3} dy \equiv \int_0^1 dy \int_x^{1+x} \frac{2y-t}{t^3} dt \equiv \int_0^1 dx \int_x^{1+x} \left(2t^{-3}x - t^{-2}\right) dt$$

$$= \int_0^1 \frac{dx}{(1+y)^2} = \int_1^2 \frac{dp}{p^2} = \left[-\frac{1}{p}\right]_1^2 = \frac{1}{2}$$