EDO UNIVERSITY IYAMHO
Department of Mathematics and Computer Science
MTH 211: Mathematical Methods I

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Description: This course is intended to give the students a thorough knowledge of differentiation and integration. This course covers topics such differentiation and integration and their applications. Mean value theorem. Taylor series. Real-valued functions of two or three variables. Partial derivatives chain rule, extrema, Lagrangian multipliers. Increments, differentials and linear approximations. Evaluation of line, integrals. Multiple integrals.

Prerequisites: Students should be familiar with the concept of differentiation and integration of a function.

Learning outcomes: At the completion of this course, students are expected to:

i. understand the concept of derivative,
ii. differentiate explicit and implicit function,
iii. understand the concept of line, double and triple integrals,
iv. understand the concept of total derivative.

Assignments: We expect to have assignments, tests and a final examination for this course. The assignments will be given at the end of a particular topic and two tests will be administered and structured as preparation for the final examination.

Grading: We will assign 10% of this class grade to assignments, 10% each for the two tests totalling 20% and 70% for the final examination. The Final examination is comprehensive.

Textbook: The recommended textbook for this class are as stated:

Title: Engineering Mathematics, 7th Edition
Authors: K. A. Stroud and Dexter J. Booth.

Title: Higher Engineering Mathematics.
Authors: Dass, H. K. and Verma, E. R.

Title: Schaum's Outline: Advanced Calculus, Fourth Edition.
Authors: Robert Wrede and Murray R. Spiegel.

Real Valued Function

Peter Dirichlet a German Mathematician (1829) conceived a function as a variable, called the dependent variable having its value fixed or determine in some definite manner by the value assigned to the independent variable x or to several independent variables $x_1, x_2, \ldots, x_n$. The value of both y and x are real. The statement $y = f(x)$ is read as y is a function of x. Again, $y = f(x_1, x_2, \ldots, x_n)$ indicates the interdependence between the variable y and x. The function $f(x)$ is usually given as an explicit formula such as $f(x) = x^2 - 3x + 5$ for all x real.

Usually, in algebraic expression, a real variable x may take any value in a certain range. If the lowest value of x is **a** and the highest value of x is **b** and x may take any value between **a** and **b** , then x is said to be a continuous variable in the range $[\mathbf{a}, \mathbf{b}]$ and takes all values such that $\mathbf{a} \leq x \leq \mathbf{b}$. Since the end points are included among the values of x which form this range, the interval is called a closed interval. The interval defined by the inequality $\mathbf{a} < x < b$ is called an open interval and is denoted by $(\mathbf{a}, \mathbf{b})$.

Review of Differentiation

1. Function of a Function.

If y is a function of u and that u itself is a function of x, then the derivative of y with respect to x is

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$

This is also called the chain rule of differentiation.

<u>Example1</u>: Find the derivative of each of the following.

(a) $y = (3x^2 - 2)^3$ (b) $y = \sqrt{(1 - 2x^3)}$

$(c)\ \ y = \dfrac{5}{(6 - \mathrm{x}^2)^3}$ $\qquad$ $(d)\ \ y = \dfrac{1}{\sqrt{(1 + \mathrm{x}^2)}}$

Solution

(a) $y = (3\mathrm{x}^2 - 2)^3$

Let $u = 3\mathrm{x}^2 - 2\ ,\ y = u^3$

$$\therefore \frac{du}{d\mathrm{x}} = 6\mathrm{x}, \quad \frac{dy}{du} = 3u^2$$

$$\frac{dy}{d\mathrm{x}} = \frac{dy}{du} \times \frac{du}{d\mathrm{x}} = 3u^2 \times 6\mathrm{x} = 18\mathrm{x}u^2 = 18\mathrm{x}(3\mathrm{x}^2 - 2)^2$$

(b) $y = \sqrt{(1 - 2\mathrm{x}^3)}$

Let $u = 1 - 2\mathrm{x}^3\ ,\ y = u^{1/2}$

$$\therefore \frac{du}{d\mathrm{x}} = -6\mathrm{x}^2, \quad \frac{dy}{du} = \frac{1}{2}u^{-1/2} = \frac{1}{2u^{1/2}}$$

$$\frac{dy}{d\mathrm{x}} = \frac{dy}{du} \times \frac{du}{d\mathrm{x}} = \frac{1}{2u^{1/2}} \times -6\mathrm{x}^2 = \frac{-6\mathrm{x}^2}{2u^{1/2}} = \frac{-3\mathrm{x}^2}{\sqrt{u}} = \frac{-3\mathrm{x}^2}{\sqrt{(1 - 2\mathrm{x}^3)}}$$

(c) $y = \dfrac{5}{(6 - \mathrm{x}^2)^3}$

Let $u = 6 - \mathrm{x}^2\ ,\qquad y = \dfrac{5}{u^3} = 5u^{-3}$

$$\therefore \frac{du}{d\mathrm{x}} = -2\mathrm{x}\ , \quad \frac{dy}{du} = -15u^{-4} = \frac{-15}{u^4}$$

$$\frac{dy}{d\mathrm{x}} = \frac{dy}{du} \times \frac{du}{d\mathrm{x}} = \frac{-15}{u^4} \times -2\mathrm{x} = \frac{30\mathrm{x}}{u^4} = \frac{30\mathrm{x}}{(6 - \mathrm{x}^2)^4}$$

(d) $y = \dfrac{1}{\sqrt{(1 + \mathrm{x}^2)}}$

Let $u = 1 + \mathrm{x}^2\ ,\qquad y = \dfrac{1}{u^{1/2}} = u^{-1/2}$

$$\therefore \frac{du}{dx} = 2x \ , \quad \frac{dy}{du} = -\frac{1}{2}u^{-3/2} = \frac{-1}{2u^{3/2}}$$

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx} = \frac{-1}{2u^{3/2}} \times 2x = \frac{-x}{u^{3/2}} = \frac{-x}{(1+x^2)^{3/2}} = \frac{-x}{\sqrt{(1+x^2)^3}}$$

2. Derivative of a Product.

If $y = uv$ where u and v are functions of x, then the derivative of y with respect to x is

$$\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$$

<u>Example2</u>: Find the derivative of each of the following.

(a) $y = (3+2x)(1-x)$ (b) $y = (1-2x+3x^2)(4-5x^2)$

(c) $y = \sqrt{x}\,(1+2x)^2$ (d) $y = (5x^4+1)^7 sin3x$

Solution

(a) $y = (3+2x)(1-x)$

Let $u = 3+2x\ ,\ v = 1-x$

$$\frac{du}{dx} = 2, \quad \frac{dv}{dx} = -1$$

$$\therefore \frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx} = (3+2x)(-1) + (1-x)(2) = -3-2x+2-2x$$
$$= -1-4x$$

(b) $y = (1-2x+3x^2)(4-5x^2)$

Let $u = 1-2x+3x^2\ ,\ v = 4-5x^2$

$$\frac{du}{dx} = -2+6x\ , \quad \frac{dv}{dx} = -10x$$

$$\therefore \frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx} = (1 - 2x + 3x^2)(-10x) + (4 - 5x^2)(-2 + 6x)$$
$$= -10x(1 - 2x + 3x^2) + (6x - 2)(4 - 5x^2)$$

(c) $y = \sqrt{x}\,(1 + 2x)^2$

Let $u = \sqrt{x} = x^{1/2}$, $v = (1 + 2x)^2$

$$\frac{du}{dx} = \frac{1}{2x^{1/2}}, \quad \frac{dv}{dx} = 4(1 + 2x)$$

$$\therefore \frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx} = x^{1/2}(4 + 8x) + (1 + 2x)^2\left(\frac{1}{2x^{1/2}}\right)$$
$$= x^{1/2}(4 + 8x) + \frac{(1 + 2x)^2}{2x^{1/2}}$$
$$= \frac{2x(4 + 8x) + (1 + 2x)^2}{2x^{1/2}}$$

(d) $y = (5x^4 + 1)^7 sin3x$

Let $u = (5x^4 + 1)^7$, $\quad v = sin3x$

$$\frac{du}{dx} = 7(5x^4 + 1)^6(20x^3), \quad \frac{dv}{dx} = 3cos3x$$

$$\therefore \frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx} = (5x^4 + 1)^7(3cos3x) + sin3x(140x^3(5x^4 + 1)^6)$$
$$= 3cos3x(5x^4 + 1)^7 + 140x^3 sin3x(5x^4 + 1)^6$$

3. Derivative of a Quotient.

If $y = \frac{u}{v}$ where u and v are functions of x such that $v \neq 0$ then the derivative of y with respect to x is

$$\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$

<u>**Example3**</u>**:** Find the derivative of each of the following.

(a) $y = \dfrac{1 + x^2}{1 - x^2}$ $\qquad$ (b) $y = \dfrac{3 + 2x - x^2}{\sqrt{1 + x}}$

$(c)\ y = \dfrac{2 + x}{x^2 + 2x + 7}$ $\qquad$ $(d)\quad y = \dfrac{\sqrt[3]{(1 + 3x^2)^2}}{x}$

Solution

(a) $\quad y = \dfrac{1 + x^2}{1 - x^2}$

Let $u = 1 + x^2, \qquad v = 1 - x^2$

$$\frac{du}{dx} = 2x\,, \quad \frac{dv}{dx} = -2x$$

$$\therefore \frac{dy}{dx} = \frac{v\dfrac{du}{dx} - u\dfrac{dv}{dx}}{v^2} = \frac{(1 - x^2)2x - (1 + x^2)(-2x)}{(1 - x^2)^2}$$

$$= \frac{(2x - 2x^3) + (2x + 2x^3)}{(1 - x^2)^2} = \frac{4x}{(1 - x^2)^2}$$

(b) $\quad y = \dfrac{3 + 2x - x^2}{\sqrt{1 + x}}$

Let $u = 3 + 2x - x^2, \qquad v = \sqrt{1 + x} = (1 + x)^{1/2}$

$$\frac{du}{dx} = 2 - 2x\,, \quad \frac{dv}{dx} = \frac{1}{2(1 + x)^{1/2}}$$

$$\therefore \frac{dy}{dx} = \frac{v\dfrac{du}{dx} - u\dfrac{dv}{dx}}{v^2} = \frac{\left((1 + x)^{1/2}\right)2(1 - x) - (3 + 2x - x^2)\left(\dfrac{1}{2(1 + x)^{1/2}}\right)}{\left((1 + x)^{1/2}\right)^2}$$

$$= \frac{2\left((1 + x)^{1/2}\right)(1 - x) - \left(\dfrac{3 + 2x - x^2}{2(1 + x)^{1/2}}\right)}{1 + x}$$

$$= \frac{\dfrac{4(1 + x)(1 - x) - (3 + 2x - x^2)}{2(1 + x)^{1/2}}}{1 + x}$$

$$= \frac{4 - 4x^2 - 3 - 2x + x^2}{2(1 + x)^{3/2}}$$

$$= \frac{1 - 2x - 3x^2}{2(1 + x)^{3/2}} = \frac{1 - 2x - 3x^2}{2\left(\sqrt{(1 + x)^3}\right)}$$

(c) $y = \dfrac{2 + x}{x^2 + 2x + 7}$

Let $u = 2 + x, \qquad v = x^2 + 2x + 7$

$\dfrac{du}{dx} = 1\,, \quad \dfrac{dv}{dx} = 2x + 2$

$$\therefore \frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2} = \frac{(x^2 + 2x + 7) - (2 + x)(2x + 2)}{(x^2 + 2x + 7)^2}$$

$$= \frac{x^2 + 2x + 7 - 2x^2 - 6x - 4}{(x^2 + 2x + 7)^2}$$

$$= \frac{-x^2 - 4x + 3}{(x^2 + 2x + 7)^2} = \frac{3 - 4x - x^2}{(x^2 + 2x + 7)^2}$$

(d) $y = \dfrac{\sqrt[3]{(1 + 3x^2)^2}}{x}$

Let $u = \sqrt[3]{(1 + 3x^2)^2} = (1 + 3x^2)^{2/3}, \qquad v = x$

$\dfrac{du}{dx} = \dfrac{2}{3}(1 + 3x^2)^{-1/3}6x = \dfrac{4x}{(1 + 3x^2)^{1/3}}, \quad \dfrac{dv}{dx} = 1$

$$\therefore \frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2} = \frac{\frac{4x^2}{(1 + 3x^2)^{1/3}} - (1 + 3x^2)^{2/3}}{x^2}$$

$$= \frac{\frac{4x^2 - 1 - 3x^2}{(1 + 3x^2)^{1/3}}}{x^2} = \frac{x^2 - 1}{x^2(1 + 3x^2)^{1/3}}$$

$$= \frac{x^2 - 1}{x^2\left(\sqrt[3]{(1 + 3x^2)}\right)}$$

Implicit Differentiation

The process of differentiating implicit function is called implicit differentiation.

<u>Example4</u>: Differentiate the following implicitly.

(a) $x^2 + y^2 = 4$ (b) $x^2y + xy^2 + 4x = 1$

Solution

(a) $x^2 + y^2 = 4$

Differentiating term by term with respect to x we have

$$2x + 2y\frac{dy}{dx} = 0$$

$$2y\frac{dy}{dx} = -2x$$

$$\frac{dy}{dx} = \frac{-2x}{2y}$$

$$\frac{dy}{dx} = \frac{-x}{y}$$

(b) $x^2y + xy^2 + 4x = 1$

$$2xy + x^2\frac{dy}{dx} + y^2 + 2xy\frac{dy}{dx} + 4 = 0$$

$$x^2\frac{dy}{dx} + 2xy\frac{dy}{dx} = -2xy - y^2 - 4$$

$$\frac{dy}{dx}(x^2 + 2xy) = -2xy - y^2 - 4$$

$$\frac{dy}{dx} = \frac{-2xy - y^2 - 4}{x^2 + 2xy}$$

Assignment

(a) $4xy^2 - 5x^2y^3 + 4y = 0$ (b) $(x + y)^2 = 5$ (c) $3x^2 + 3xy^2 - cos2y = 0$

Parametric Equation

Given that $x = \sin t$ and $y = \cos t$ where t is a parameter are called parametric equation.

Example5: Given that $x = 5t^3$ and $y = 4t^2$, find $\frac{dy}{dx}$

Solution

$x = 5t^3$

$$\frac{dx}{dt} = 15t^2 \quad \rightarrow \frac{dt}{dx} = \frac{1}{15t^2}$$

$y = 4t^2$

$$\frac{dy}{dt} = 8t$$

$$\therefore \frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx} = 8t \times \frac{1}{15t^2} = \frac{8}{15t}$$

Higher Derivatives

Given that $y = f(x)$, $\frac{dy}{dx}$ is also a function of x. The derivative of $\frac{dy}{dx}$ with respect to x is $\frac{d}{dx}\left(\frac{dy}{dx}\right)$, $\frac{d}{dx}\left(\frac{dy}{dx}\right)$ is called the second derivative of y with respect to x and is usually denoted with $\frac{d^2y}{dx^2}$.

The third derivative is $\frac{d^3y}{dx^3}$ while the fourth derivative is $\frac{d^4y}{dx^4}$. In general $\frac{d^ny}{dx^n}$ is the nth derivative of y with respect to x.

Example6: Find the first, second and third derivatives of the following.

(a) $y = 3x^4$ $\quad (b)\ y = e^{x^4}$ $\quad (c)\ y = ln x$ $\quad (d)\ y = sin 3x$

Solution

(a) $y = 3x^4$

$$\frac{dy}{dx} = 12x^3$$

$$\frac{d^2y}{dx^2} = 36x^2$$

$$\frac{d^3y}{dx^3} = 72x$$

(b) $y = e^{x^4}$

$$\frac{dy}{dx} = 4x^3e^{x^4}$$

$$\frac{d^2y}{dx^2} = 12x^2.e^{x^4} + 4x^3.4x^3e^{x^4} = 12x^2e^{x^4} + 16x^6e^{x^4}$$

$$\frac{d^3y}{dx^3} = 24x.e^{x^4} + 12x^2.4x^3e^{x^4} + 96x^5.e^{x^4} + 16x^6.4x^3e^{x^4}$$

$$= 24xe^{x^4} + 48x^5e^{x^4} + 96x^5e^{x^4} + 64x^9e^{x^4}$$

(c) $y = lnx$

$$\frac{dy}{dx} = \frac{1}{x}$$

$$\frac{d^2y}{dx^2} = \frac{-1}{x^2}$$

$$\frac{d^3y}{dx^3} = \frac{2}{x^3}$$

Assignment

1. Find the first, second and third derivatives of $y = sin3x$.
2. Given that $y = acoskx + bsinkx$, show that

   $$\frac{d^2y}{dx^2} + k^2y = 0$$

3. Given that $y = e^x - e^{-x}$, show that

   $$\frac{d^3y}{dx^3} + \frac{d^2y}{dx^2} + \frac{dy}{dx} + y = 4e^x$$

Applications of Differential Calculus

1. Tangents and Normals to Curves.

At any point on a curve, $\frac{dy}{dx}$ at that point gives the gradient of the tangent at the point. A straight line perpendicular to the tangent at the point of contact of the tangent to the curve is called a normal to the curve.

__**Example1**__: Find the equation of the tangent and the normal to the curve

$y = 2x^3 - x^2 + 3x + 1$ at the point $x = 1$.

Solution

$y = 2x^3 - x^2 + 3x + 1$

$\frac{dy}{dx} = 6x^2 - 2x + 3$

$$\frac{dy}{dx}_{x=1} = 6(1)^2 - 2(1) + 3 = 7$$

If m is the gradient of the tangent at the point $x = 1$, then $m = 7$

At the point $x = 1$, we have that $y = 2 - 1 + 3 + 1 = 5$

The equation of the tangent at the point $x = 1$ is given by

$$y - y_1 = m(x - x_1)$$

$y - 5 = 7(x - 1)$

$y - 5 = 7x - 7$

$y - 7x + 2 = 0$

If m' is the gradient of the normal at $x = 1$, then

$$m' = \frac{-1}{m} = \frac{-1}{7}$$

Hence, the equation of the normal at the point $x = 1$ is

$y - y_1 = \frac{-1}{m}(x - x_1)$

$$y - 5 = \frac{-1}{7}(x - 1)$$

$$7(y - 5) = -1(x - 1)$$

$$7y - 35 = 1 - x$$

$$x + 7y - 36 = 0$$

Example2: Find the equation of the tangent to the curve $x^2y + xy^3 + 3x - 13 = 0$

At the point $(1, 2)$

Solution

$$x^2y + xy^3 + 3x - 13 = 0$$

$$2xy + x^2\frac{dy}{dx} + y^3 + 3xy^2\frac{dy}{dx} + 3 = 0$$

$$x^2\frac{dy}{dx} + 3xy^2\frac{dy}{dx} = -2xy - y^3 - 3$$

$$(x^2 + 3xy^2)\frac{dy}{dx} = -2xy - y^3 - 3$$

$$\frac{dy}{dx} = \frac{-2xy - y^3 - 3}{x^2 + 3xy^2}$$

$$\frac{dy}{dx}_{x=1,\ y=2} = \frac{-2xy - y^3 - 3}{x^2 + 3xy^2} = \frac{-4 - 8 - 3}{1 + 12} = \frac{-15}{13}$$

If m is the gradient of the tangent at the point $x = 1,\ y = 2,$ then $m = \frac{-15}{13}$. Therefore the equation of the tangent at the point $x = 1,\ y = 2$ is given by

$$y - y_1 = m(x - x_1)$$

$$y - 2 = \frac{-15}{13}(x - 1)$$

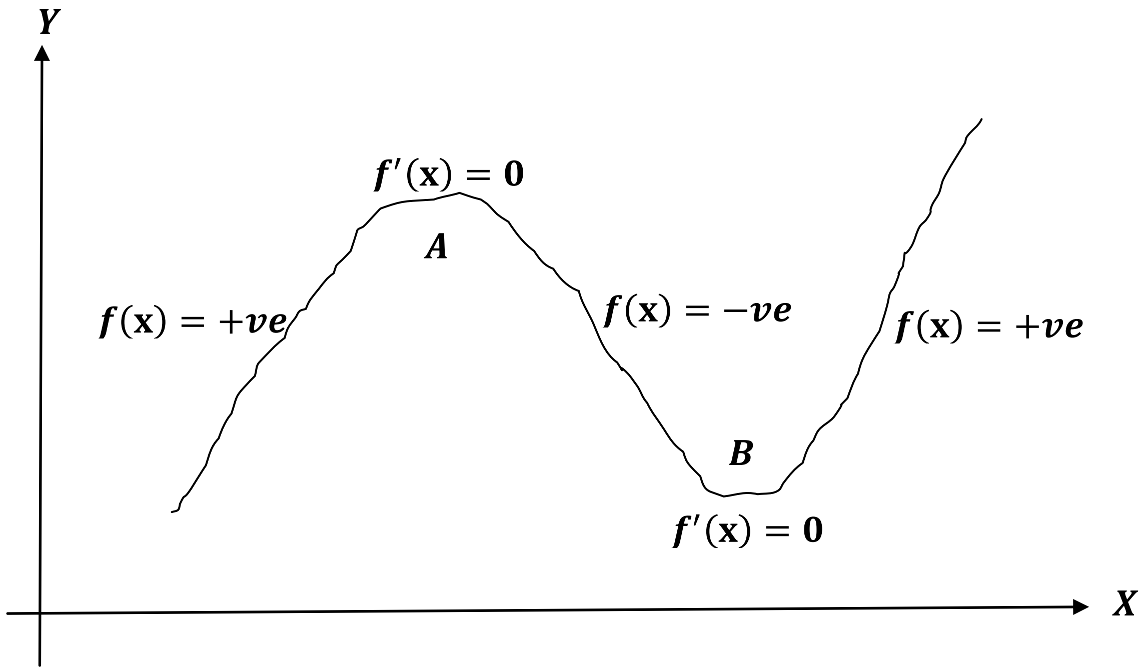
$$13(y - 2) = -15(x - 1)$$

$$13y - 26 = -15x + 15$$

$15x + 13y = 41$

$15x + 13y - 41 = 0$

2. Maximum and Minimum Points.



The point **A** in the figure above where the gradient is changing from positive through zero to negative is called the maximum point. The point **B** where the gradient is changing from negative through zero to positive is called a minimum point. The maximum and the minimum points are both called turning points. The point where $\boldsymbol{f'(x) = 0}$ is called a stationary point. The value of y at the maximum point is called the maximum value while the minimum point of y is called the minimum value.

To find the turning points of a given function, we shall carry out the following test

$$(a)\ \frac{dy}{dx} = 0, \qquad \frac{d^2y}{dx^2} < 0 \quad \text{(Maximum point)}$$

$$(b)\ \frac{dy}{dx} = 0, \qquad \frac{d^2y}{dx^2} > 0 \quad \text{(Minimum point)}$$

Example1: Find and classify the turning points of the following curves

1. $y = x^2 - 5x + 6$
2. $y = x^2 + 4x - 3$

Solution

1. $y = x^2 - 5x + 6$

$$\frac{dy}{dx} = 2x - 5$$

$$\frac{dy}{dx} = 0$$

$$2x - 5 = 0$$

$$x = \frac{5}{2}$$

$$\frac{d^2y}{dx^2} = 2$$

This shows that it is minimum point.

2. $y = x^2 + 4x - 3$

$$\frac{dy}{dx} = 2x + 4$$

$$\frac{dy}{dx} = 0$$

$$2x + 4 = 0$$

$$x = -2$$

$$\frac{d^2y}{dx^2} = 2$$

This shows that it is minimum point.

Assignment

Find the turning points of the following curve $y = x^3 - 12x + 6$.

Review of Integration

Differentiation measures the rate of change while integration measures area.

If $\frac{dy}{dx} = f(x)$, then $dy = dx f(x)$

$$\int dy = \int dx f(x)$$

$$y = F(x)dx + C$$

where C is called an arbitrary constant of integration.

<u>**Example1**</u>: Evaluate the following integrals

$(a)\quad \int (x^7 + 1)dx \qquad (b)\quad \int \sqrt{x}dx \qquad (c)\quad \int (2x^2 + 3x + 8)dx$

Solution

$$(a)\quad \int (x^7 + 1)dx = \int x^7 dx + \int 1dx$$

$$= \frac{x^8}{8} + x + C$$

$$(b)\quad \int \sqrt{x}dx = \int x^{1/2}dx$$

$$= \frac{2x^{3/2}}{3} + C$$

$$(c)\quad \int (2x^2 + 3x + 8)dx = \int 2x^2 dx + \int 3x dx + \int 8dx$$

$$= \frac{2x^3}{3} + \frac{3x^2}{2} + 8x + C$$

Techniques of Integration

1. The Integral of the form

$$\int f(a\mathrm{x}+b)d\mathrm{x}$$

Let $U = a\mathrm{x}+b$

$$\frac{d\mathrm{u}}{d\mathrm{x}} = a$$

$$\frac{d\mathrm{x}}{d\mathrm{u}} = \frac{1}{a}$$

$$d\mathrm{x} = \frac{1}{a}d\mathrm{u}$$

$$\therefore \int f(a\mathrm{x}+b)d\mathrm{x} = \int f(u)\frac{1}{a}du = \frac{1}{a}\int f(u)du.$$

<u>Example2</u>: Evaluate the following integrals

(a) $\int (2\mathrm{x}+3)^5 d\mathrm{x}$ (b) $\int \frac{3d\mathrm{x}}{(4\mathrm{x}-1)^3}$ (c) $\int e^{-5\mathrm{x}+2}d\mathrm{x}$

Solution

(a) $\int (2\mathrm{x}+3)^5 d\mathrm{x}$

Let $U = 2\mathrm{x}+3$

$$\frac{d\mathrm{u}}{d\mathrm{x}} = 2$$

$$\frac{d\mathrm{x}}{d\mathrm{u}} = \frac{1}{2}$$

$$d\mathrm{x} = \frac{1}{2}d\mathrm{u}$$

$$\int (2\mathrm{x}+3)^5 d\mathrm{x} = \int U^5\frac{1}{2}d\mathrm{u}$$

$$= \frac{1}{2}\int U^5 d\mathrm{u} = \frac{1}{12}U^6 + C = \frac{1}{12}(2\mathrm{x}+3)^6 + C$$

$(b)\quad \int \frac{3dx}{(4x-1)^3}$

Let $U = 4x - 1$

$\frac{du}{dx} = 4$

$\frac{dx}{du} = \frac{1}{4}$

$dx = \frac{1}{4}du$

$$\int \frac{3dx}{(4x-1)^3} = \int \frac{3}{U^3}\frac{1}{4}du$$

$$= \frac{3}{4}\int \frac{1}{U^3}du = \frac{3}{4}\int U^{-3}du = \frac{3}{4}\frac{U^{-2}}{-2} + C = \frac{-3}{8U^2} + C = \frac{-3}{8(4x-1)^2} + C$$

$(c)\quad \int e^{-5x+2}dx$

Let $U = -5x + 2$

$\frac{du}{dx} = -5$

$\frac{dx}{du} = \frac{-1}{5}$

$dx = \frac{-1}{5}du$

$$\int e^{-5x+2}dx = \int e^u \frac{-1}{5}du = \frac{-1}{5}\int e^u du = \frac{-1}{5}e^u + C = -\frac{1}{5}e^{-5x+2} + C$$

2. The Integral of the form

$$\int \frac{f'(x)}{f(x)}dx$$

Let $U = f(x)$

$$\frac{du}{dx} = f'(x)$$

$$du = f'(x)dx$$

$$\frac{du}{u} = \frac{f'(x)}{f(x)}dx$$

$$\int \frac{du}{u} = \int \frac{f'(x)}{f(x)}dx$$

$$\int \frac{du}{u} = \int u^{-1}du = Log_e u.$$

$$Log_e u = \int \frac{f'(x)}{f(x)}dx$$

Hence, $\int \frac{f'(x)}{f(x)}dx = Log_e f(x) + C$

<u>Example3</u>: Evaluate the following integrals

$(a) \quad \int \frac{2xdx}{x^2+1}dx \qquad\qquad (b) \quad \int \frac{3x^2+4x}{x^3+2x^2+7}dx$

Solution

$(a) \quad \int \frac{2xdx}{x^2+1}dx$

$$\frac{d}{dx}(x^2+1) = 2x$$

$$\therefore \int \frac{2xdx}{x^2+1}dx = \int \frac{\frac{d}{dx}(x^2+1)}{x^2+1}dx = Log_e(x^2+1) + C$$

$(b) \quad \int \frac{3x^2+4x}{x^3+2x^2+7}dx$

$$\frac{d}{dx}(x^3+2x^2+7) = 3x^2+4x$$

$$\therefore \int \frac{3x^2 + 4x}{x^3 + 2x^2 + 7} dx = \int \frac{\frac{d}{dx}(x^3 + 2x^2 + 7)}{(x^3 + 2x^2 + 7)} dx = Log_e(x^3 + 2x^2 + 7) + C$$

3. Integration by algebraic substitution

An integral that is not in the standard form can be reduced to one that is in the standard form by making appropriate algebraic substitution

Example4: Evaluate the following integrals

$(a) \quad \int xe^{x^2} dx \qquad\qquad (b) \quad \int x^2 cos x^3 dx$

Solution

$(a) \quad \int xe^{x^2} dx$

Let $U = x^2$

$$\frac{du}{dx} = 2x$$

$$\frac{dx}{du} = \frac{1}{2x}$$

$$dx = \frac{1}{2x} du$$

$$\therefore \int xe^{x^2} dx = \int xe^u \frac{1}{2x} du = \frac{1}{2} \int e^u du = \frac{1}{2} e^u + C = \frac{1}{2} e^{x^2} + C$$

$(b) \quad \int x^2 cos x^3 dx$

Let $U = x^3$

$$\frac{du}{dx} = 3x^2$$

$$\frac{dx}{du} = \frac{1}{3x^2}$$

$$dx = \frac{1}{3x^2} du$$

$$\therefore \int x^2 cos x^3 dx = \int x^2 cos u \frac{1}{3x^2} du = \frac{1}{3} \int cos u du = \frac{1}{3} sin u + C = \frac{1}{3} sin x^3 + C$$

Assignment

Evaluate the following integrals

$(a) \quad \int x\sqrt{1 + x^2} dx \qquad\qquad (b) \quad \int x^2 (3 + x^3)^{3/2} dx$

4. Integration by Parts

This is a technique use for integrating product were substitution method fails. If $y = uv$, then by product rule we have

$$\frac{dy}{dx} = u \frac{dv}{dx} + v \frac{du}{dx}$$

$$\therefore \int \left(\frac{dy}{dx}\right) dx = \int \left(u \frac{dv}{dx}\right) dx + \int \left(v \frac{du}{dx}\right) dx$$

$$\therefore y = \int u dv + \int v du$$

$$\therefore \int u dv = y - \int v du$$

$$\therefore \int u dv = uv - \int v du$$

<u>Example5</u>: Evaluate the following integrals

$(a) \quad \int x sin x dx \qquad\qquad (b) \quad \int x^2 cos x dx$

Solution

$(a) \quad \int x sin x dx$

Let $u = x$

$$\frac{du}{dx} = 1$$

$$dv = sin\text{x}dx$$

$$v = -cos\text{x}$$

$$\therefore \int udv = uv - \int vdu$$

$$\int xsinxdx = -xcosx - \int cosxdx$$

$$\int xsinxdx = sinx - xcosx + C$$

(b) $\int x^2 cosxdx$

Let $u = x^2$

$$\frac{du}{dx} = 2x$$

$$dv = cos\text{x}dx$$

$$v = sin\text{x}$$

$$\therefore \int udv = uv - \int vdu$$

$$\int xsinxdx = x^2 sinx - \int 2xsinxdx$$

$$= x^2 sinx - 2\int xsinxdx$$

$$= x^2 sinx - 2sinx + 2xcosx + C$$

Assignment

Evaluate the following integrals

(a) $\int x^2 lnxdx$ $\qquad$ (b) $\int e^{2x} cos2xdx$

5. Integration by Partial Fractions

If a rational expression is not in a standard integral form, it could be transformed into a standard form by splitting it into partial fractions.

Example6: Evaluate the following integrals

$(a)\quad \int \frac{4x-5}{(x+1)(x-2)}dx \qquad (b)\quad \int \frac{2x^3-2x^2-2x-7}{x^2-x-2}dx$

Solution

First split $\frac{4x-5}{(x+1)(x-2)}$ into its partial fractions.

$$\frac{4x-5}{(x+1)(x-2)} = \frac{A}{(x+1)} + \frac{B}{(x-2)}$$

$A(x-2) + B(x+1) = 4x-5$

Using cover up method, to solve for A let $x=-1$ and to solve for B let $x=2$

$A(-1-2) = 4(-1)-5$

$-3A = -9$

$A = 3$

Similarly, $B = 1$

$$\therefore \frac{4x-5}{(x+1)(x-2)} = \frac{3}{(x+1)} + \frac{1}{(x-2)}$$

$$\int \frac{4x-5}{(x+1)(x-2)}dx = \int \left(\frac{3}{(x+1)} + \frac{1}{(x-2)}\right)dx$$

$$= \int \left(\frac{3}{x+1}\right)dx + \int \left(\frac{1}{x-2}\right)dx$$

$$= 3ln(x+1) + ln(x-2) + C$$

$(b)\quad \int \frac{2x^3-2x^2-2x-7}{x^2-x-2}dx$

Since the degree of the numerator is higher than the degree of the denominator, the expression must be in a proper algebraic fraction form.

$$\begin{array}{r|l} & 2x \\ \hline x^2 - x - 2 & 2x^3 - 2x^2 - 2x - 7 \\ & \underline{2x^3 - 2x^2 - 4x} \\ & \qquad\qquad\quad 2x - 7 \end{array}$$

$$\therefore \; \frac{2x^3 - 2x^2 - 2x - 7}{x^2 - x - 2} = 2x + \frac{2x - 7}{x^2 - x - 2}$$

Splitting $\frac{2x - 7}{x^2 - x - 2}$ into partial fraction, we have

$$\frac{2x - 7}{x^2 - x - 2} = \frac{A}{(x+1)} + \frac{B}{(x-2)}$$

$$A(x - 2) + B(x + 1) = 2x - 7$$

Solving for A and B, we have $A = 3$ and $B = -1$ respectively

$$\frac{2x - 7}{x^2 - x - 2} = \frac{3}{(x+1)} - \frac{1}{(x-2)}$$

$$\therefore \; \frac{2x^3 - 2x^2 - 2x - 7}{x^2 - x - 2} = 2x + \frac{3}{(x+1)} - \frac{1}{(x-2)}$$

$$\therefore \; \int \frac{2x^3 - 2x^2 - 2x - 7}{x^2 - x - 2} dx = \int \left(2x + \frac{3}{(x+1)} - \frac{1}{(x-2)}\right) dx$$

$$= x^2 + 3\ln(x + 1) - \ln(x - 2) + C$$

Assignment

Evaluate the following integrals

(a) $\int \frac{4x - 23}{(x - 5)^2} dx$ $\qquad$ (b) $\int \frac{2}{x^2 - 4} dx$